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Billiards Periodicity in Polygons

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Declaration

I certify that this thesis, submitted for the degree of Master in Mathematics to the Department of Mathematics at Birzeit University. And that this thesis (or any part of it) has not been submitted for a higher degree to any other university.

Ruba Shahwan Signature................
February 6th, 2017

Abstract:

In this research we survey the periodicity of billiards in some class of polygons (the polygons which tiles the plane with reflection), then we study the periodicity of billiards in triangles. It has been shown that by elementary geometric ways that an acute, right and isosceles triangles always have a periodic billiard path. Schwartz write a computer assisted proof that a triangle has a periodic billiard path when all its angles are at most one hundred degree. But the conjecture that does the triangles which have an angle greater than 100 degree is still open.

Keywords: Billiards in polygon, Periodicity, McBilliards program, Translation surface.

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Preface

The study of mathematical billiard is one of the interesting subjects by the researchers nowadays. The theory of studying mathematical billiards can be partitioned into three areas: A convex billiards, billiards in polygons and dispersing billiards. A mathematical billiard consists of a domain (i.e billiard table) and a mass point (the billiard ball), that moves freely without friction according to the optic law that is the angle of incidence is equal to the angle of reflection. The idea is to reflect the polygon with respect to the side it hits, instead of reflecting the trajectory. In 1775, Fagnano proved that the combinatorial orbit 123 describes a periodic orbit on every acute triangle in [1].

An acute, right and isosceles triangles always have periodic billiard paths by elementary geometric ways: The fagnano path exists on a triangle if and only if the triangle is acute, In 1986, Masur showed that every rational polygons (i.e polygons whose angles are rational multiple of π) admits infinitely many periodic orbits which are distinct [17], and in 1987 katok proved that the number of periodic orbits of a given period grows exponentially. Pat and Hooper shows that right triangle does not have a stable periodic billiard paths [14]. Hanson and Kolan showed that every orbit which is perpendicular to the base of a right triangle is periodic [15].

Furthermore, it has been shown that the billiard inside a unit square has a periodic trajectory if the slope of a trajectory is rational, and it is everywhere dense and uniformly distributed if the slope is irrational. Regular trajectories which does not pass through the corners of the polygon. However, one can also study trajectories emitted from one corner and trapped after several reflections in some other (or the same) corner, these trajectories are called the generalized diagonals.

Why mathematicians are especially interested in rational billiards:

1. Without rationality assumption , few tools are available and not much is known.

2. With the rationality assumption a wide range of interesting behavior is possible depending on the choice of polygon.m v

3. The rationality assumption leads to surprising and beautiful connections to algebraic geometry, $Teichm\breve{u}ller$ theory , ergodic theory on homogenous spaces, and other areas of mathematicians.

This survey thesis consists mainly of three chapters. Chapter 1 consists of basic definitions and results. In chapter 2 we introduce some of the basic results in polygons especially the polygons that tile the plane with reflection and triangles. In chapter 3 we explain how the mcbilliards applet allows us to prove periodicity in an obtuse triangle whose one angle is at most 100 degree, and we study also the stability issue.

Chapter 1

Basics and preliminaries

1.1 Euler Characteristic and maniolds

Definition 1.1.1. *[16] A manifold M of dimension n, is a topological space with the following properties:*

- *M is Hausdorff.*
- *M is locally Euclidean of dimension n, and.*
- *M has a countable basis of open sets.*

Definition 1.1.2. *[26] A surface or a 2-dimensional manifold is any object, such that for any point $p \in S$, there is a small region U on the surface, which surrounds and contain the point p.*

Definition 1.1.3. *[6] The number of handles is called the genus of M and is denoted by g.*

Definition 1.1.4. *[26] A triangulation is cutting a surface X into a finite number of 'polygonal' regions called faces, by smooth non-self-intersecting arcs, called edges, joined at vertices.*

Definition 1.1.5. *[26] Two surfaces are called* Homeomorphic *if one of them can be triangulated, then cut along a subset of the edges into pieces, and then glue back together according to the instructions given by the orientation and labels on the edges, in order to obtain the other surface.*

Definition 1.1.6. *[5] The Euler characteristic of a finitely triangulated surface is $\chi = V - E + F$ where V is the number of vertices, E denotes the number of edges and F is the number of faces.*

Example 1. *[26] Consider the cube in three dimensions. The cube has eight vertices, twelve edges, and six faces. So the Euler characteristic of the cube is*

$$\begin{aligned}\chi &= V - E + F\\ &= 8 - 12 + 6\\ &= 2\end{aligned}$$

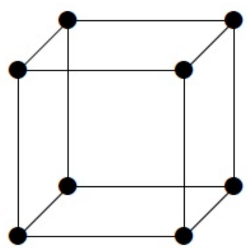

Figure 1.1: Cube in three dimension.

Prposition 1. *[26] The Euler characteristic of a surface does not depend on its triangulation i.e the Euler characteristic does not change if we subdivide a face or an edge of a polyhedron.*

Proof: Let P be an n-gon.

- If we add a single vertex, then the number of vertices will be changed by 1, then we must connect this vertex with the other vertices, this will result another n edges, so $\triangle E = n$. The resulting edges will divide the face into n subfaces. So the number of faces increases by $n - 1$, so the total change in Euler characteristic is

$$\begin{aligned}\Delta\chi &= \Delta V - \Delta E + \Delta F\\ &= 1 - n + n - 1\\ &= 0\end{aligned}$$

- Suppose that we subdivide an edge by adding a new vertex in the middle of the edge, so the number of vertices is increased by one, and the number of edges increased by one. But the number of faces remains the same, therefore the total change of the characteristic is

$$\Delta\chi_E = 1 - 1 + 0 = 0$$

Definition 1.1.7. *[6] The standard n-simplex, denoted by σ^n, is the subset of $\mathbb{R}^{n+1}$ given by*

$$\sigma^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \quad, \forall i, \sum_{i=0}^{n} x_i = 1\}.$$

The 0-simplex is a point and the 1-simplex is an interval while the 2-simplex is a triangle.

Definition 1.1.8. *[6] A triangulation of surface M is orientable if its 2-simplices admits a coherent collection of orienations and non-orientable otherwise.*

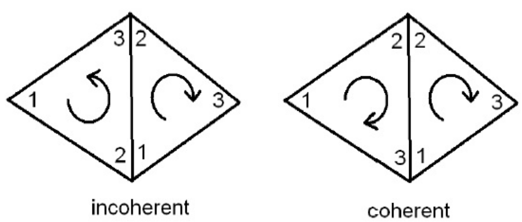

Figure 1.2: Coherent and incoherent orientations of 2-simplices.

Remark 1. *[6] The Euler characteristic of the orientable surface of genus g is $2 - 2g$.*

Proof: By induction on g

- If g = 0, then $\chi_0 = 2 - 2(0) = 2$, and the orientable surface is a sphere with Euler characteristic 2 [6].
- Assume that the assumption is true for g = n i.e $\chi_n = 2 - 2n$.

- For g = n + 1, then $\chi_{n+1} = 2 - 2(n+1) = 2 - 2n - 2 = \chi_n - 2$. But the Euler characteristic decreases by 2 when the number of handles increases by one [6]. Hence by induction the assumption is true ■.

Definition 1.1.9. *[14] The greatest common divisor (gcd) of a set of positive integers $\{a_1, ..., a_k\}$ is the largest integer which divides every element of the set denoted by $gcd(a_1, ..., a_k)$.*

Definition 1.1.10. *[16] A chart for M is a homeomorphism $\phi : U \to V$ where U is an open set in M and V is an open set in $\mathbb{R}^n$. A chart for M is denoted by (U, ϕ).*

Definition 1.1.11. *[16] A collection of charts $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha | \alpha \in \mathcal{I}\}$ is called an atlas for M if $\bigcup_{\alpha \in \mathcal{I}} U_\alpha = M$.*

Definition 1.1.12. *[16] If $\mathcal{A}$ and $\mathcal{B}$ are two differentiable atlases then their union is also differentiable atlas. Equivalentely, for every chart ϕ in $\mathcal{A}$ and ψ in $\mathcal{B}$, $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are smooth.*

Definition 1.1.13. *[16] Two atlases A_1, A_2 of M are called **equivalent** if and only if $A_1 \cup A_2$ is an atlas of M. An equivalence class of atlases in M is called conformal structure. A maximal atlas of M is the union of all atlases in a conformal structure.*

Definition 1.1.14. *[3] M is called a differentiable n-manifold if the following hold:*

- *M_1 : The set M is covered by a collection of charts, that is every point is represented in at least one chart.*
- *M_2 : has an atlas; that is M can be written as a union of compatible charts.*

Definition 1.1.15. *[6] Let ϕ_α, ϕ_β be two charts, then the map between the two charts*

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi(u_\alpha \cap u_\beta) \to \phi_\alpha(u_\alpha \cap u_\beta)$$

is called the transition map.

Definition 1.1.16. *[3] A Riemann surface is a two-dimensional, connected, Hausdorff topological manifold M with a countable base for the topology and with conformal transition maps between charts i.e, there exists a family of* *open set $\{u_\alpha\}$ covering M and homeomorphisms ϕ_α: $u_\alpha \to v_\alpha$ where $v_\alpha \subset \mathbb{R}^2$ is some open set so that*

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi(u_\alpha \cap u_\beta) \to \phi_\alpha(u_\alpha \cap u_\beta)$$

is biholomorphic .

Definition 1.1.17. *[3] A function f on a manifold M into N is said to be smooth if for every $p \in U$ there is a chart (U, ϕ) for M and a chart (V, ψ) for N at a point $f(p)$ with $f(U) \subseteq V$ such that the partial derivatives of*

$$\psi \circ f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$$

exists and are continuous to all orders, that is ,$\psi \circ f \circ \phi^{-1}$ is smooth .

1.1.1 Examples of Riemann manifolds

Example 2. *[18] Any open region Δ of the complex plane. Let (Δ, z) where z is the identity map on Δ to be the associated chart. And let (U, ϕ) with U is an open subset of Δ, and $\phi : U \to \mathbb{C}$ is biholomorphic.*

Example 3. *[18] The sphere $\mathbb{S}^2$ can be made into the Riemann sphere by equipping it with the complex structure*

The Riemann sphere $S^2 \subset \mathbb{R}^3$, which can be described in three conformally equivalent ways : $S^2, \mathbb{C}_+\infty, \mathbb{C}P^1$.

- Define the conformal structure on S^2 via two charts

  $$(S^2 \setminus (0,0,1), \phi_+), (S^2 \setminus (0,0,-1), \phi_-)$$

  where $\phi_\pm$ are the stereographic projections

  $$\phi_+(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}, \phi_-(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3}$$

  From north and south pole, respectively If

  $$P = (x_1, x_2, x_3) \in S^2 \ \ with \ \ x_3 \neq \pm 1$$

  , then if $p = (x_1, x_2, x_3) \in S^2$ and $x_3 \neq \pm 1$, then we have

  $$\phi(p)_+ \phi(p)_- = (\frac{x_1 + ix_2}{1 - x_3})(\frac{x_1 - ix_2}{1 + x_3}) = \frac{x_1^2 + x_2^2}{1 - x_3^2}$$

$$= \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} = 1$$

so the transition map between these two charts equals $z \to \frac{1}{z}$.

- The one point compactification of $\mathbb{C}$ which is denoted by $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. And define the neighborhood base of ∞ in $\mathbb{C}_\infty$ is given by the complements of all compact sets of $\mathbb{C}$. Define the corresponding two charts by $(\mathbb{C}, \mathbb{Z}), (\mathbb{C}_\infty \backslash \{0\}, \frac{1}{z})$. Clearly the transition map between these charts is $z \to \frac{1}{z}$

- The one-dimensional complex projective space

$$\mathbb{C}P^1 = \{[z : w]/(z, w) \in \mathbb{C}^2 \setminus \{0, 0\}/ \sim$$

  where the equivalence relation is $(z_1, w_1) \sim (z_2, w_2)$ if and only if $z_2 = \lambda z_1, w_2 = \lambda w_1$ for some $\lambda \in \mathbb{C}$ Our charts are (U_1, ϕ_1) and (U_2, ϕ_2) where

$$U_1 = \{[z, w] \in \mathbb{C}P^1/w \neq 0\}, \phi_1[z : w]\} = \frac{z}{w}$$
$$U_2 = \{[z : w] \in \mathbb{C}P^1/z \neq 0\}, \phi_2([z : w]) = \frac{w}{z}$$

  The transition map here is also $z \to \frac{1}{z}$.

Example 4. *[18] Any smooth orientable two-dimensional submanifold of* $\mathbb{R}^3$ *is a Riemann manifold .*

Definition 1.1.18. *[3] The holomorphic functions on a Riemann surface M are defined as all analytic functions $f : M \to \mathbb{C}_\infty$. They are denoted by $\mathcal{H}(M)$.*

Chapter 2

Billiards in polygons

In this chapter we introduce in the first section some basic definitions and results in polygonal billiards, then in second section we recall the translation covers in triangular billiard surfaces; the main results of this section will be found in [19]. Harris shows in [11] that the polygons which tile the plane with reflection have a periodic billiard orbit. Moreover they have a perpendicular billiard orbit.

2.1 Basic definitions and results of billiards in polygons

Definition 2.1.1. *[11] A polygonal Billiard table is any closed bounded by a convex polygon in the Euclidean plane with a point inside the polygon that has an initial position and direction associated with it.*

Definition 2.1.2. *[11] The angle of incidence is the acute angle between an incoming ray and the tangent line, if it exists, of a boundary where the ray strikes.*

Definition 2.1.3. *[27] A labeling of an n-gon is a bijection between the set of edges of the n-gon to the set* $\mathbb{Z}/n\mathbb{Z} = \{1, 2, ..., n\}$*, where adjacent edges lie above edges sharing a vertex are sent to adjacent numbers in* $\mathbb{Z}/n\mathbb{Z}$*. A polygon with a labeling is called a labeled polygon.*

Definition 2.1.4. *[11] Given an incoming ray and outgoing ray (with a vertex on the boundary) the angle of reflection is the acute angle created from the exiting ray and the line tangent to where the ray struck the boundary.*

Definition 2.1.5. *[11] The unfolding when the billiard ball hits the boundary of a polygon instead of reflecting the motion of the ball we can reflect the polygon about the edge it hits, and then allow the billiard ball to go straight through. In this way we keep reflecting the polygon and the billiard path will be a straight line through this sequence of polygons. This sequence of polygons form an unfolding.*

Definition 2.1.6. *[11] Given a polygonal billiard table, a billiard orbit is the curve obtained from the path of a point inside the billiard table moving in its given direction where whenever it strikes a boundary it reflects of the boundary and the angle of incidence is congruent to its corresponding angle of reflection if the orbit strikes a vertex on the polygon the orbit ends.*

Definition 2.1.7. *[11] A periodic orbit is an orbit in which the point returns to its initial position with the same initial direction.*

Definition 2.1.8. *[11] A perpendicular periodic orbit is a periodic orbit with some angle of incidence equal to* $\pi/2$.

Definition 2.1.9. *[11] For an n-gon* $Z \in \mathbb{R}^2$ *and any billiard path* $\{s_i\}$ *in it, let* w_i *be the label of the edge containing the ending point of* s_i*, for each* $i \in y\mathbb{Z}$*. Then* $\{w_i\}$ *is a sequence of labels . This is called the orbit type of the billiard path .*

Definition 2.1.10. *[27] Let* $\{s_i\}$ *be a billiard path in Z with corresponding unfolding* $\{Z_i\}$*. Then each* Z_j *has a corresponding billiard path* $\{s_{i,j}\}$*. Then the unfolding representation of* s_i *is the union* $L = \cup_{i\in\mathbb{Z}} s_{i,i}$*, which would be a straight line contained in the unfolding domain D.*

Definition 2.1.11. *[27] The space of labeled n-gons is* $\tilde{Z}_n = \{$*n-gons in* $\mathbb{R}^2$ *with a labeling* $\}$.

Definition 2.1.12. *[12] A translation surface is a surface built by identifying edges of polygons. Two edges may be identified if they are only parallel, have the same length and opposite orientations.*

Prposition 2. *[27]* $\tilde{Z}_n$ *can be seen as open subset of* $\mathbb{R}^{2n}$.

Proof: Define the map $f : \tilde{Z}_n \to \mathbb{R}^{2n}$ such that for each labeled polygon T, and suppose that $v_i \in \mathbb{R}^2$ be the vertex between edges i and edges i+1, then $f(Z) = (v_1, v_2, ..., v_n) \in \mathbb{R}^{2n}$. Since each polygon is determined by its vertices and the ordering of its vertices its labeling, this map is injective, so $\tilde{Z}_n$ can be seen as a subset of $\mathbb{R}^{2n}$.
Let the collection of n vertices $v = (v_1, ..., v_n)$ form a labeled polygon and let $r = \frac{1}{2} min|v_i - v_j|$. Let B_i be the open ball in $\mathbb{R}^2$ with center v_i and with radius r, and consider the open set $F = \Pi B_i$. For any $p = (p_1, ..., p_n) \in F$, then for all i, $p_i \in B_i$, define the polygonal curve γ by joining $p_1 p_2, ..., p_n p_1$ which is a closed piecewise linear curve. Suppose $v_i v_{i+1}$ and $v_j v_{j+1}$ are disjoint line segments, then the vertices of these two line segments will be at least $2r$ unit apart, so points on one of the line segments will be at least ■.

Definition 2.1.13. *[19] A flow is called topologically transitive if it has a dense orbit.*

Definition 2.1.14. *[9] A periodic trajectory in a triangle is stable if for any small perturbation of the triangle, the triangle obtained contains a periodic trajectory close to the initial one.*

Definition 2.1.15. *[9] Two trajectories are called* ***close****, if they have the same number of reflection pints and the corresponding reflection points lies on the same edge of the triangle, close to each other.*

Definition 2.1.16. *[9] An orbit is uniformly distributed if the amount of time that it spends in a region is proportional to the area of the area of the region.*

Lemma 2.1.1. *[24] Let the angles of a billiard k-gon be $\frac{\pi m_i}{n_i}$, $i = 1, ..., k$, where m_i and n_i are coprime, and N be the least common multiple of $n_i' s$. Then genus $M = 1 + \frac{N}{2}(k - 2 - \Sigma \frac{1}{n_i})$.*

Proof: Let the ith vertex V with the angle $\frac{\pi m_i}{n_i}$ and define the group obtained by reflections in the sides of Z adjacent to V by G_i, this group of linear transformation contains $2n_i$ elements of linear transformations of the plane generated by the reflections in the sides of Z adjacent to V by G_i. Then G_i composed of $2n_i$ elements.

The number of the copies of the polygons Z_i. According to the construction of M.

- the number of copies of the polygons Z_j that are glued. together at V equals the cardinality of the orbit of the test angle θ under the group G_i that is equals $2n_i$. We had 2N copies of the polygon Z, and therefore 2N copies of the vertex V; after the gluing we have $\frac{N}{n_i}$ copies of this vertex on the surface M.
- The total number of V in M equals $N\Sigma\frac{1}{n_i}$.
- The total number of edges equals Nk, and the number of faces is 2N.

If we substitute these in the Euler characteristic we get

$$N\sum\frac{1}{n_i} - Nk + 2N = 2 - 2g$$

where g is the genus

$$2g = 2 - 2N + Nk - N\sum\frac{1}{n_i}$$

$$2g = 2 + N(2 + k - \frac{1}{\sum n_i})$$

$$g = 1 + \frac{N}{2}(k - 2 - \sum\frac{1}{n_i})$$

■.

Prposition 3. *[27] In a polygon Z, if two billiard paths $\{s_j\}$ and $\{l_j\}$ have the same orbit type, then s_j is parallel to l_j for all $j \in \mathbb{Z}$*

Proof: *Suppose that $\{w_j\}$ is the orbit type, denote the corresponding unfolding by $\{Z_j\}$ and the unfolding domain by D. Without loss of generality we can rotate and translate everything so that the unfolding representation of $\{s_j\}$ is the straight line so that it coincide with the x - axis and suppose that $L \subset D$ be the unfolding representation of $\{l_j\}$.*

For every j, Z_j is the same polygons with the same size, therefore let $d > 0$ be the diameter of Z_j for all j.

Claim 2.1.1. *L is horizontal*

Proof: *By contradiction, assume that L is not horizontal then it can be parametrized by $x = ay + b$ for some $a, b \in \mathbb{R}$. Then we can find a point (x_0, y_0) on L with $y_0 > d$ and $x_0 = ay_0 + b$. Then while $L \in D$, $(x_0, y_0) \in Z_j$ for some $j \in \mathbb{Z}$. However as Z_j has diameter d and it intersects with the positive x-axis, all points in Z_j will have y coordinate less than or equal to d, contradiction to our claim. So L must be parallel to the x-axis* ■.

Prposition 4. *[27] In a polygon Z, a billiard path is periodic if and only if its orbit type is periodic.*

Proof: *Let $\{s_j\}$ be any billiard path with the corresponding periodic orbit type $\{w_j\}$ with minimal period h.*

- *If h is even define k to be equal to h.*
- *if h is odd let k = 2h, so in both cases, k is always even .*

Let $\{Z_j\}$ be the unfolding corresponding to $\{w_j\}$ and let D be the unfolding domain, and L be the unfolding representation of $\{s_j\}$.

For each Z_j, let c_i be its centroid. If k is even, then Z_0 and Z_k will have the same orientation, i.e we can obtain Z_k from Z_0 by translation and rotation. Let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation and $r : \mathbb{R}^2 \to \mathbb{R}^2$be the rotation centered on c_k such that $f = r \circ t$ will send Z_0 to Z_k, so by periodicity, f^n would send Z_0 to Z_{nk}.

Connect the points $...., c_{-2k}, c_{-k}, c_0, c_k, c_{2k}, c_{3k}, ...$ by line segments. Let r be a rotation that is not a multiple of 2π, then $\angle c_0 c_k c_{2k}$ is not a multiple of 2π. Then the points c_0, c_k, c_{2k} will determine a circle S (if $\angle c_0 c_k c_{2k}$ is an odd multiple of 2π. $c_0$and c_{2k} will coincide, so S will be the circle with diameter $c_0 c_k$). Since f is a rigid motion(i.e a motion which preserves distance), and $f^n(c_0 c_k) = c_{nk} c_{(n+1)k}$ and $f^n(\angle c_0 c_k c_{2k}) = \angle c_{nk} c_{(n+1)k} c_{(n+2)k}$. c_{nk} lies on R for all $n \in \mathbb{Z}$. Let 0 be the center of S, and define $d = sup\{|h - 0|, h \in \cup_{j \in \mathbb{Z}} Z_j\}$, which exists as $\cup_{j=0}^{k} Z_j$ is compact. Then by periodicity we have $d = sup\{|p - 0| : p \in \cup_{j \in \mathbb{Z}} Z_j = D\}$. Let S^1 be the closed ball centered at 0 and with radius d, then $D \subset S^1$ must be bounded, while the straight line L is contained in D and cannot be bounded, so we get a contradiction so the rotation r must be a multiple of 2π, so Z_0 and Z_k differ only by translation .

Now s_0 ends in the edge w_0 and s_k ends in the edge $w_k = w_0$ with the same direction. Rotate and scale everything so that the edge w_0 of Z is the line segment from $(0,0)$ to $(1,0)$. Suppose s_0 ends in $(a,0)$ and s_k ends in (b,0), and suppose a does not equal to b. Without loss of generality assume $b > a$. Then by periodicity of the unfolding, s_{2k} would ends in $(b+(b-a),0)$, and s_{nk} would ends in $(a+n(b-a),0)$. For n large enough, we would have $a+n(b-a) > 1$, then s_{nk} will be out of the polygon Z_{nk}. So we get a contradiction, so a must be equal to b, therefore $s_0$and s_k would be line segment in Z ending in the same spot with the same direction, so the billiard path would repeat itself ever since, so the billiard path is periodic ■.

Corollary 1. *[27] If a billiard path is periodic, let $\{Z_j\}$ be the corresponding unfolding, and let h be the minimal even period. Then Z_h can be obtained from Z_0 by a translation and this translation is in the direction of the unfolding representation L of the billiard path.*

Theorem 2.1.2. *[25] The billiard in a typical polygon is topologically transitive.*

Theorem 2.1.3. *[25] The billiard in a typical polygon is ergodic.*

Theorem 2.1.4. *[24] In an isosceles triangle ABC with right angle B, there is no billiard path from A coming back to A.*

Proof: *If we unfold the isosceles right triangle ABC as illustrated in the figure below. The vertices labelled A which are the images of the vertex A*

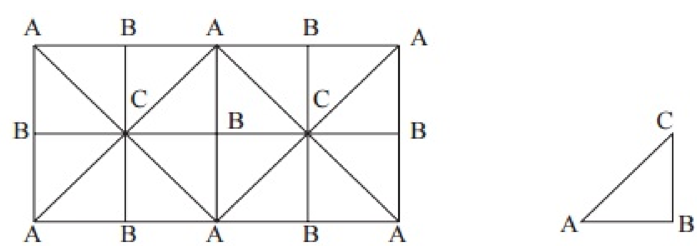

Figure 2.1: The unfolding of right isosceles triangle.

of the triangle have both coordinates even, But the vertices labelled B and C at least one of their coordinates must be odd. Assume not, that there exist a billiard trajectory from A back to A. Then the corresponding unfolding line *joining the vertex* $(0,0)$ *to a vertex* $(2l,2q)$*. This line will pass through* (l,q) *which may be the B vertex or C vertex or both* l *and* q *are even. And then the line passes through the point* $(\frac{l}{2},\frac{q}{2})$ *and so on.*

2.2 Translation covers among triangular billiard surfaces

In 2015 Schmurr [19] discussed all translation covers among triangular billiard surfaces that arise from a particular case of billiards in a triangle whose interior angles are all rational multiples of π. In this section we will consider some of the corresponding results.

2.2.1 The rational billiards construction

Define R *to be a rational polygonal billiard region i.e the interior angles are all rational multiple of* π*. Let* D_{2Q} *be the dihedral group of order* $2Q$ *generated by Euclidean reflections in the sides of* R *if the particle moves with this region at constant speed and with initial direction vector* v*, changing direction when it reflects off the sides of* R*, moves according to the optic law (i.e the angle of incidence is equal to the angle of reflection). Every subsequent direction vector for the particle is of the form* $\delta.v$*, for some element* $\delta \in D_{2Q}$ *where* D_{2Q} *acts on* $\mathbb{R}^2$ *via Euclidean reflections.* $j = 1, ..., r$ *where* m_j, n_j *are coprime positive integers and* r *is the number of vertices then the group* $G(Q)$ *is isomorphic to the dihedral group* D_N*, where* N *is the least common multiple of* $n_1, n_2, ..., n_r$*. Consider the set* $D_{2Q}.R$ *of* $2Q$ *copies of* R *transformed by the elements of* D_{2Q}*. For each edge* e *of* R*, we consider the corresponding elements* $\rho_e \in D_{2Q}$ *which represents reflection across* e*. For each* $\delta \in D_{2Q}$ *, we glue* $\rho_e\delta.R$ *and* $\delta.R$ *together a long their copies of* e*. The result is a closed Riemann surface with flat structure induced by the tiling by* $2Q$ *copies of* R*.*

This surface is an example of a compact translation surface. A compact translation surface can be defined as the result of gluing together a finite set of polygons in the plane a long parallel edges in such a way that the result is a compact surface. Equivalently, a translation surface can be defined as a real two manifold.

Definition 2.2.1. *[19]* ***A conical singularity p on a flat structure*** *is a point such that, in the flat metric induced by the coordinate maps, the total angle (" cone angle ") about p is not equal to* 2π.

Definition 2.2.2. *[19] Let X be a flat surface with conical singularities. Let* $\tilde{X}$ *be the flat surface obtained by puncturing all singularities of X. If all transition functions of* $\tilde{X}$ *are translations, then X is a translation surface.*

Definition 2.2.3. *[19] The cone angle of conical singularities on the translation surface are always integer multiple of* 2π.

Theorem 2.2.1. *[21] Every rational polygons has a periodic billiard path.*

2.2.2 Billiards in rational triangle

Let (a_1, a_2, a_3) *be a triple of positive integers. Let* $T(a_1, a_2, a_3)$ *denotes a triangle with internal angles* $\frac{a_1\pi}{Q}$, $\frac{a_2\pi}{Q}$ *and* $\frac{a_3\pi}{Q}$ *(where* $Q = a_1 + a_2 + a_3$ *and gcd* $(a_1, a_2, a_3) = 1$*) suppose that* $X(a_1, a_2, a_3)$ *be a translation surface a rising from billiards in* $T(a_1, a_2, a_3)$ *via the Fox-Kershner [19]. This surface is called a triangular billiard surface. If the triangle is isosceles or right, the corresponding surface is called an isosceles or right triangular billiard surface.*

Definition 2.2.4. *[19] A billiard triangulation is a triangulation* τ *of X whose triangles are the various elements of* $D_{2Q}.T$.

Remark 2. *[19] Let* $T(a_1, a_2, a_3)$*, label the vertices of T as* v_1, v_2 *, and* v_3*, where* v_i *corresponds to* a_i*. Hence the total number of triangles in* τ *is* $2Q$*, that the number of vertices of* τ *corresponding to* v_i *is* $gcd(a_i, Q)$ *and each member of this set has a cone angle of* $(\frac{a_i}{gcd(a_i,Q)})2\pi$.

Definition 2.2.5. *[19] Let* v_1, v_2, v_3 *be the vertices of* $T(a_1, a_2, a_3)$ *and let* $\pi_X : X(a_1, a_2, a_3) \to T(a_1, a_2, a_3)$ *be the standard projection. A vertex class of* $X(a_1, a_2, a_3)$ *is any of three sets* $\pi_X^{-1}(v_1)$, $\pi_X^{-1}(v_2)$*, or* $\pi_X^{-1}(v_3)$*, for a given vertex class either all the elements are singular or all are nonsingular; hence we call a vertex class singular if its elements are singularities and nonsingular if its elements are nonsingular.*

A vertex class $\pi_X^{-1}(v_i)$ *is nonsingular if and only if* a_i *divides* Q*. Moreover, the sum of cone angles of the elements of* $\pi_X^{-1}(v_i)$ *is* $2a_i\pi$.

Definition 2.2.6. *[19] A translation cover is a holomorphic (possibly ramified) cover of translation surfaces $f : X \to Y$ such that, for each pair of coordinate maps ϕ_X and ϕ_Y on X and Y, respectively, the map $\phi_Y \circ f \circ \phi_X^{-1}$ is a translation when ϕ_X and ϕ_y are restricted to open sets not containing singular points, we say f is balanced if f does not mapping points to nonsingular points.*

Definition 2.2.7. *[19] We say that X and Y are translation equivalent if there exists a degree 1 translation cover $f : X \to Y$.*

Lemma 2.2.2. *[19] Suppose f: $X(a_1, a_2, a_3) \to X(b_1, b_2, b_3)$ is a translation cover of triangular billiard surfaces. Let $\pi_X : X(a_1, a_2, a_3) \to T(a_1, a_2, a_3)$ and $\pi_Y : X(b_1, b_2, b_3) \to T(b_1, b_2, b_3)$ be the canonical projections to triangles with vertices v_1, v_2, v_3 and w_1, w_2, w_3 respectively. Suppose that $p \in \pi_Y^{-1}(w_i)$, $p' \in \pi_X^{-1}(v_j)$ and $f(p') = p$ with a ramification index of m at p'. Then*

$$\frac{mb_i}{gcd(b_i, b_1 + b_2 + b_3)} = \frac{a_j}{gcd(a_j, a_1 + a_2 + a_3)}$$

Proof: The cone angle of $\acute{P}$ is m times the cone angle at p. So the result follows from remark 2.

Lemma 2.2.3. *[19] Let a and b be relatively prime positive integers not both equal to one. The right triangular billiard surface $Y = X(a_1 + a_2, a_1, a_2)$ is related to two isosceles triangular billiard surfaces X_1 and X_2 via translation covers $f_i : X_i \to Y$ for each i, if a_i is odd then $X_i = X(2a_j, a_i, a_i)$ and f_i has degree 2, if it is even then $X_i = X(a_j, \frac{a_i}{2}, \frac{a_i}{2})$ and f_i has degree one.*

There are only three surfaces $X(1, 1, 2), X(1, 2, 3)$and$X(1, 1, 1)$ such that these surfaces are triangular billiard surfaces which have no singularities, and the only three triangular surfaces of genus 1; furthermore $X(1, 2, 3)$ and $X(1, 1, 1)$ are actually translation equivalent. Each of these surfaces admits balanced translation covers of itself by itself of arbitrary high degree, this fact is related to the fact that $T(1, 1, 2)$, $T(1, 2, 3)$ and $T(1, 1, 1)$ are the only Euclidean triangles which tile the Euclidean plane by flips. Since about ramification points flat ramified covers are locally of the form $z \to z^{\frac{1}{n}}$ for some $n > 1$, implying that the cone angle of the ramification point is greater than 2π, hence ramification points are singular.
Each trajectory in a rational polygon can take at most $2M$ different directions

after all successive collisions. This stands in contrast to the dynamics in an irrational billiard where the number of directions explored by a single trajectory is infinite. Identifying the corresponding opposite sides of R one gets a surface of genus g, where

$$g = 1 + \frac{M}{2}\Sigma_{i=1}^{i=k}\frac{n_i - 1}{m_i}.$$

The genus equals to unity for rectangle, equilateral triangle and triangle with angles equal to $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{6}$ and $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{4}$, polygon R has in these cases the shape of rectangle, parallelogram or regular hexagon and is (after identification of the opposite sides) topologically equivalent to a torus .

A simple example of genus 2 the rhombus with vertex angle equal to $\frac{\pi}{3}$ or in general a $\frac{\pi}{3}$ parallelogram, the triangle with the angles ($\frac{2\pi}{3}$,$\frac{\pi}{6}$, $\frac{\pi}{6}$), ($\frac{3\pi}{5}$, $\frac{\pi}{5}$, $\frac{\pi}{5}$) , ($\frac{\pi}{2}$, $\frac{3\pi}{8}$, $\frac{\pi}{8}$) , the deltoids with angles ($\frac{2\pi}{3}$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{\pi}{3}$) and ($\frac{3\pi}{4}$, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{\pi}{4}$) or the trapeze ($\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$)

The genus 3 is characteristic for $\frac{\pi}{4}$ parallelogram or for a billiard in the shape of 3-rectangular steps. Also the triangles ($\frac{3\pi}{8}$, $\frac{3\pi}{8}$, $\frac{\pi}{4}$), ($\frac{\pi}{7}$, $\frac{\pi}{7}$, $\frac{5\pi}{4}$) , ($\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$) , ($\frac{\pi}{3}$, $\frac{\pi}{9}$, $\frac{5\pi}{9}$), a rectangle with the right triangle cut a way a long one side or the trapeze ($\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, $\frac{\pi}{6}$) leads to dynamics of three holes.

Some examples of genus four billiards like $\frac{\pi}{5}$ rhombus, $\frac{2\pi}{3}$ hexagon and $\frac{\pi}{3}$.

The rhombus with vertex angle equal to $\frac{\pi}{(N+1)}$ or the N-steps rectangular staircase correspond to manifolds of genus g =N. knowing examples of billiards belonging to each of the classes labeled by the finite genus g.

2.3 Polygons that tile the plane with reflections

In 2007 Harris proved that in [11] the class of polygons that tile the plane with reflection admits a periodic orbit. Moreover this class admits a perpendicular periodic orbit. In this section we recall these results.

Definition 2.3.1. *[11] A polygon tiles the plane with reflections if given any sequence of polygons x_n that is generated by reflection across one side of the proceeding polygon, then for all $m, n \in \mathbb{N}$, the intersection of the interiors of polygon x_n and x_m is either empty or the full interior of x_n.*

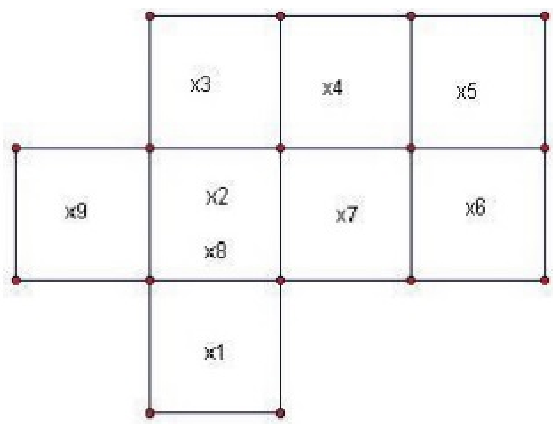

Figure 2.2: An example of a polygon that tile the plane with reflection.

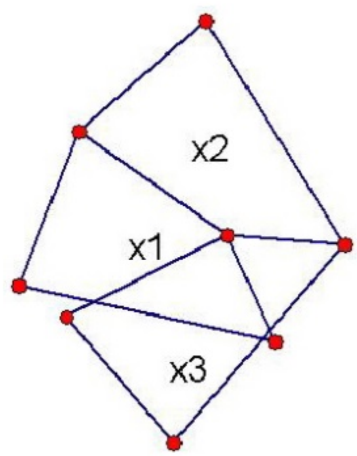

Figure 2.3: An example of a polygon that does not tile the plane with reflection.

Lemma 2.3.1. *[11] The sum of interior angles in a polygon is given by* $\sum_{i=1}^{s} a_i = (s-2)\pi = \frac{s-2}{2}2\pi$ *where* s *is the number of sides in the polygon and* a_i *is an interior angle .*

Theorem 2.3.2. *[11] A polygon that tiles the plane with reflections must have angles in the form* $\alpha_i = \frac{2\pi}{k_i}$ *where* α_i *is any angle and* $k_i \in \mathbb{Z}$ *and* $k_i \geq 3$.

Proof: The polygon that tile the plane must be completely covered with that polygon with no overlaps. The angles around each vertex must be equal since they are obtained by reflections. The sum of the angles around each vertex is 2π. So this vertex must be of the form $k_i\alpha_i = 2\pi$, therefore $\alpha_i = \frac{2\pi}{k_i}$. Here k_i represents the number of angles around the vertex, k_i cannot be of less than three, since if that happens, then the point will not be a vertex ■.

Theorem 2.3.3. *[11] A polygon that tiles the plane with reflection has between three and six sides.*

Proof: The sides of the polygon can not be less than three sides. So a polygon that tile the plane must have at least three sides. By lemma 2.3.1 and theorem 2.3.2 we can conclude that the polygon must have three or more sides.

$$\sum_{i=1}^{s} \frac{2\pi}{k_i} = \sum_{i=1}^{s} a_i = (s-2)\pi = \frac{(s-2)}{2}2\pi.$$

So

$$\sum_{i=1}^{s} \frac{1}{k_i} = \frac{(s-2)}{2} \tag{2.1}$$

since $k_i \geq 3$ then $\sum_{i=1}^{s} \frac{1}{k_i} \leq \frac{s}{3}$. If $s > 6$, then

$$\frac{s}{3} = \frac{2s}{6} = \frac{3s-s}{6} < \frac{3s-6}{6} = \frac{s-2}{2}$$

So if $s > 6$ then

$$\sum_{i=1}^{n} \frac{1}{k_i} \leq \frac{s}{3} < \frac{s-2}{2}$$

and so the equality $\sum_{i=1}^{s} \frac{1}{k_i} = \frac{s-2}{2}$ in (2.1) will not hold.

So a polygon that tiles the plane with reflections has between $3 \leq s \leq 6$ ■.

Theorem 2.3.4. *[11] The class of a billiard tables that tile the plane with reflections is [the regular hexagons; the rectangle; the rhombus with angles* $2\pi/3$ *and* $\pi/6$ *; any kite with angles* $2\pi/3, \pi/2, \pi/2, and \pi/3$*; any triangle with angles* $2\pi/3$ *and two angles of* $\pi/6$ *; any triangle with one right angle, one angle of* $\pi/3$ *and one angle of* $\pi/6$*; any triangle with a right angle and two angles of* $\pi/4$ *; and any triangle with all angles* $\pi/3$*]*

Proof:

Case 1 Six sides. Let $b_1, b_2, ..., b_6$ be the angles of a hexagon without loss of generality we write our angles as $b_1 \geq b_2 \geq ...... \geq b_6$

$$\sum_{i=1}^{6} b_i = (6-2)\pi = 4\pi$$

$$\sum_{i=1}^{6} b_i = \sum_{i=1}^{6} \frac{2\pi}{k_i} = 4\pi.$$

So,

$$\sum_{i=1}^{6} \frac{2\pi}{k_i} = 4\pi.$$

If we divide both sides by 2π we obtain

$$\sum_{i=1}^{6} \frac{1}{k_i} = 2$$

because of how are angles are organized we have

$$\frac{1}{k_1} \geq \frac{1}{k_2} \geq ..... \geq \frac{1}{k_6}.$$

So,

$$k_1 \leq k_2 \leq ...... \leq k_6.$$

Let $k_1 = 4$, then

$$4 \leq k_i \quad ;for \quad every \quad i$$

$$\frac{1}{k_i} \leq \frac{1}{4} \quad ;for \quad every \quad i$$

$$\Sigma_{i=1}^{6}\frac{1}{k_i} \leq \Sigma_{i=1}^{6}\frac{1}{4} = \frac{6}{4} \neq 2.$$

So k_1 cannot be any number greater than or equal to 4. So $k_1 = 3$, then all the $k's$ must be 3. Since

$$\Sigma_{i=1}^{6}\frac{1}{3} = \frac{6}{3} = 2.$$

If we change one of the $\frac{1}{3}'s$ to a less value, then the some will not be 3. Therefore, $k_i = 3$ for all the angles and all angles are $\frac{2\pi}{3}$. So the regular hexagon is the only hexagon that tiles the plane with reflections. Because if every angle is $\frac{2\pi}{3}$, then to construct that angle the segments must be congruent otherwise the polygon will not tile the plane with reflection. Figure 2.4 explains the case If m and l are not equivalent. If that happens then after a reflection across m then another reflection across l' shows that m' and l will make give the interior of whatever polygon this set of segments belongs to be different, therefore the polygon will not tile the plane with reflections

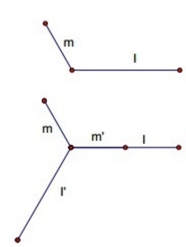

Figure 2.4: m and l are not equivalent, so the corresponding polygon will not tile the plane with reflection.

case 2 Five sides.

Let $b_1, ..., b_5$ be the sides of the pentagon, without loss of generality assume

$$b_1 \geq b_2 \geq ..... \geq b_5$$

The sides of the pentagon add up to

$$\sum_{i=1}^{5} b_i = \frac{(5-2)2\pi}{2} = \frac{3 \times 2\pi}{2}$$

$$\sum_{i=1}^{5} \frac{1}{k_i} = \frac{3}{2}.$$

Assume $k_1 = 4$

$$\sum_{i=1}^{5} \frac{1}{k_i} \leq \frac{5}{4} \neq \frac{3}{2}.$$

So $k_1 = 3$. If we take $k_2 = 4$ then at most we find

$$\frac{1}{3} + \sum_{i=2}^{5} \frac{1}{k_i} \leq \frac{4}{4} + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3} \neq \frac{3}{2}$$

So $k_2 = 3$. If we take $k_3 = 4$ then at most we have

$$\frac{1}{3} + \frac{1}{3} + \sum_{i=3}^{5} \frac{1}{k_i} \leq \frac{1}{3} + \frac{1}{3} + \frac{3}{4} = \frac{17}{4} \neq \frac{3}{2}$$

So take $k_3 = 3$, if we assume $k_4 = 5$

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{k_i} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{5} = \frac{21}{15} \neq \frac{3}{2}$$

So k_4 is either 4 or 3 .

- case 2a : $k_4 = 4$.
  If $k_4 = 4$ then k_5 has only one solution. That is

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{k_5} = \frac{3}{2}$$

$$\frac{1}{k_5} = \frac{3}{2} - \frac{1}{4} - 1 = \frac{6-5}{4} = \frac{1}{4}.$$

  So there is only one possibility for a pentagon that tiles the plane so far. Which is the pentagon with angles $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$

- case 2b : If $k_4 = 3$ then,

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{k_5} = \frac{3}{2}$$

$$\frac{1}{k_5} = \frac{3}{2} - \frac{4}{3} = \frac{9-8}{6} = \frac{1}{6}$$

$$k_5 = 6$$

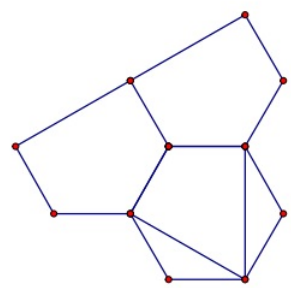

Figure 2.5: A pentagon with angles $(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2})$

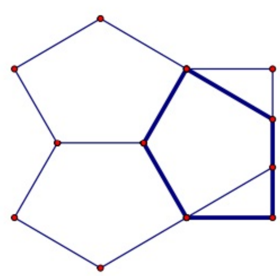

Figure 2.6: A pentagon $(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2})$

case 3: Four sided. Let b_1, b_2, b_3, b_4 be the sides of the four sided polygon such that

$$b_1 \geq b_2 \geq b_3 \geq b_4$$

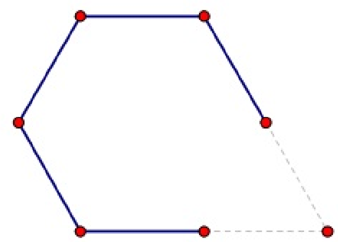

Figure 2.7: A pentagon with angles $(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3})$

$$\sum_{i=1}^{4} b_i = 2\pi$$

$$\sum_{i=1}^{4} \frac{1}{k_i} = 1$$

since

$$k_1 \leq k_2 \leq k_3 \leq k_4.$$

If $k_1 = 5$ then

$$\sum_{i=1}^{4} \frac{1}{k_i} \geq \frac{4}{5} \neq 1$$

So k_1 must be 4 or 3

- case 3a :$k_1 = 4$, then

$$\Sigma_{i=1}^{4} \frac{1}{k_i} = \frac{1}{4} + \Sigma_{i=2}^{4} \frac{1}{k_i}$$

$$\leq \frac{1}{4} + \Sigma_{4i=2}^{4} \frac{1}{4}$$

$$= \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$$

then all the $k_i's$ equals four also. This quadrilateral is a rectangle.

- case 3b : $k_1 = 3$
  If $k_2 = 5$, then

$$\sum_{i=1}^{4} \frac{1}{k_i} = \frac{1}{3} + \Sigma_{i=2}^{4} \frac{1}{k_i} \geq \frac{1}{3} + \frac{3}{5} = \frac{14}{15} \neq 1$$

  So k_2 does not equal 5, it must be either 3 or 4

  - Case3b(i): $k_2 = 4$
    If $k_3 = 5$ then

$$\begin{aligned}\sum_{i=1}^{4} \frac{1}{k_i} &\geq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3} \\ &= \frac{2}{3} + \frac{1}{4} + \frac{1}{5} \\ &= \frac{40+29}{60} \neq 1\end{aligned}$$

    so $k_3 = 3$.
    When $k_1 = 3$, $k_2 = 4$ and $k_3 = 4$,

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{k_4} = 1$$

$$\frac{5}{6} + \frac{1}{k_4} = 1$$

$$\frac{1}{k_4} = \frac{1}{6}.$$

    So that k_4 equals 6. So we obtain another quadrilateral that tiles the plane with reflection, which have angles $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}$. Such that the segments making an angle of $\frac{2\pi}{3}$ are equivalent.

    A kite is a quadrilateral where adjacent sides are congruent

  - Case 3b(ii): $k_2 = 3$, if $k_3 = 7$, then

$$\Sigma_{i=1}^{4} \frac{1}{k_i} = \frac{2}{3} + \Sigma_{i=3}^{4} \frac{1}{k_i}$$

$$> \frac{2}{3} + \frac{2}{7} = \frac{14}{21} + \frac{6}{20} = \frac{20}{21} \neq 1$$

    So when $k_1 = 3$, $k_2 = 3$, k_3 is 3,4,5 or 6.

When $k_3 = 6$, $k_4 = 6$, then the corresponding quadrilateral have angles $\frac{2\pi}{3}$, $\frac{2\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{3}$. The rhombus of these angles is the polygon that tile the plane with reflection.

If $k_3 = 5$, then $k_4 = \frac{15}{2}$. Since k_4 is not an integer so we ignore this case.

When $k_3 = 4$ then

$$\frac{2}{3} + \frac{1}{4} + \frac{1}{k_4} = 1$$
$$\frac{11}{12} + \frac{1}{k_4} = 1$$
$$\frac{1}{k_4} = \frac{1}{12}$$
$$k_4 = 12.$$

It has been shown that the corresponding polygon can not tile the plane since it is not closed.

If $k_3 = 3$ then

$$\frac{2}{3} + \frac{1}{3} + \frac{1}{k_4} = 1$$
$$\frac{1}{k_4} = 0$$

Since k_4 is not a positive integer, so this case is not eligible

- **Case 3:** Three sides.

Let b_1, b_2 *and* b_3 be the angles of the triangle and without loss of generality let

$$b_1 \geq b_2 \geq b_3.$$

So,

$$b_1 + b_2 + b_3 = \pi$$
$$\sum_{i=1}^{3} b_i = \sum_{i=1}^{3} \frac{2\pi}{k_i} = \pi$$

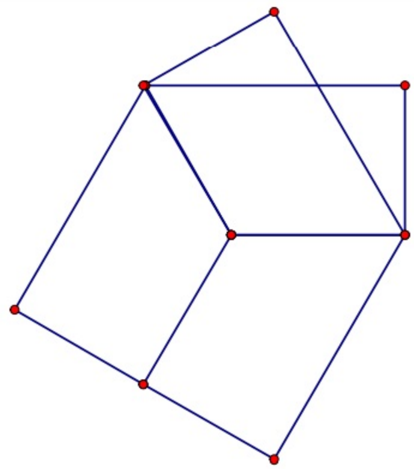

Figure 2.8: A quadrilateral with angles of the order $(\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3})$

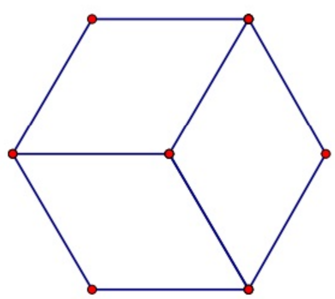

Figure 2.9: A rhombus with angles in order $(\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2})$

After dividing by 2π we have

$$\sum_{i=1}^{3} \frac{1}{k_i} = \frac{1}{2}.$$

Since

$$k_1 \leq k_2 \leq k_3$$

Assume that $k_1 = 7$, then

$$\sum_{i=1}^{3} \frac{1}{k_i} = \frac{1}{7} + \Sigma_{i=2}^{3} \frac{1}{k_i}$$

$$\leq \frac{1}{7} + \Sigma_{i=2}^{3} \frac{1}{7}$$

$$= \frac{1}{7} + \frac{2}{7}$$

$$= \frac{3}{7} < \frac{1}{2}$$

therefore, $k_1 \leq 6$

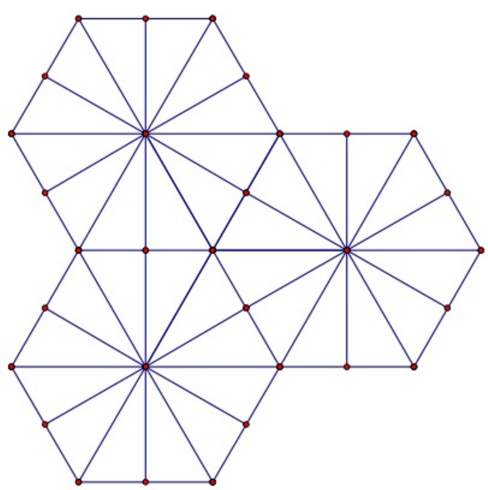

Figure 2.10: A triangle with angles of the order $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$

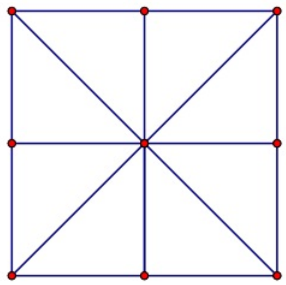

Figure 2.11: A triangle with angles of the order $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$

1. If $k_1 = 6$. Let $k_2 = 6$ then

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{k_3} = \frac{1}{2}$$

$$\frac{1}{k_3} = \frac{1}{2} - \frac{1}{3}$$

$$\frac{1}{k_3} = \frac{1}{6}.$$

   So $k_3 = 6$ which is the equilateral triangle. This is the only case that satisfies $k_1 \leq k_2 \leq k_3$ and $\Sigma_{i=1}^{3} \frac{1}{k_i} = \frac{1}{2}$.

2. $k_1 = 5$, $k_2 = 5$ and $k_3 = 10$ is the only case satisfying the above conditions. Which is the triangle which have the angles $\frac{2\pi}{5}$, $\frac{2\pi}{5}$ and $\frac{\pi}{5}$. When sketching this case the reflections are overlap. So there is no triangle that tile the plane with reflection .

3. $k_1 = 4$.

   If $k_2 = 4$, then $\frac{1}{4} + \frac{1}{4} + \frac{1}{k_3} = \frac{1}{2}$ $\frac{1}{k_3} = 0$, there is no value for k_3, k_2 must be greater than 4. If $k_2 = 5$, then $k_3 = 20$. If $k_2 = 6$, then $k_3 = 12$. If $k_2 = 7$, then $k_3 = \frac{28}{3}$ is a non integer value. If $k_2 = 8$, then $k_3 = 8$. If we sketch these cases we find overlaps only when $k_2 = 5$.

4. $k_1 = 3$. Using the conditions

$$k_1 \leq k_2 \leq k_3 \quad k_i \in \mathbb{Z}$$

and

$$\Sigma_{i=1}^{3} \frac{1}{k_i} = \frac{1}{2}$$

If k_2 is 3,4 or 5 then the summation will be greater than $\frac{1}{2}$. And if $k_2 = 6$, then $\frac{1}{3} + \frac{1}{6} + \frac{1}{k_3} = \frac{1}{2}$, k_3 has no possible value, so k_2 is greater than six.

(a) When $k_2 = 7$ then $k_3 = 42$. But if we sketch this triangle with reflection it is overlaps.

(b) When $k_2 = 8$, then $k_3 = 24$ which is also overlaps.

(c) If $k_2 = 9$, then $k_3 = 18$, this triangle overlaps by reflection.

(d) If $k_2 = 10$, then $k_3 = 15$. If we sketch this triangle it is also overlaps.

(e) If $k_2 = 11$, then there is a non integer value for k_3.

(f) If $k_2 = 12$, then $k_3 = 12$. When sketching the other possible triangles with reflection, they do not overlap since they can be reflected to obtain a hexagon■.

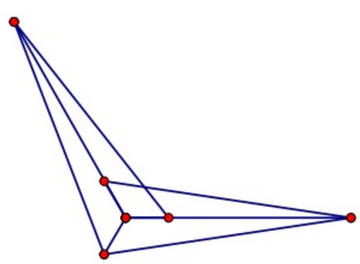

Figure 2.12: A triangle with angles of the order $(\frac{2\pi}{3}, \frac{2\pi}{7}, \frac{\pi}{21})$.

.

Theorem 2.3.5. *[11] Any periodic orbit in the class of polygons that tile the plane with reflections belongs to a strip (neighborhood) of periodic orbits.*

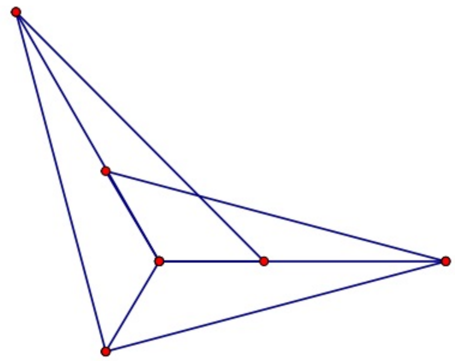

Figure 2.13: A triangle with angles of the order $(\frac{2\pi}{3}, \frac{\pi}{4}, \frac{\pi}{12})$.

Proof: Define the periodic orbit by unfolding, then there is a finite number of vertices in the corridor which is obtained by the reflection to point which is identified with the initial position. Take the distances between every segment perpendicular to the vertices and the line that represents the orbit. Label these distances by $(x_1, x_2, ..., x_n)$ and let N denotes the minimum of these distances, where $N \in \mathbb{R}^2$, N can not be equal to zero since if that happened then the orbit would not periodic (since that means the orbit will pass through one of the vertices of the polygon).

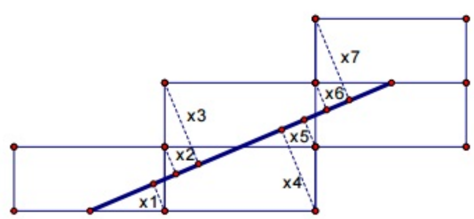

Figure 2.14: An example illustrates the distance between the vertices and periodic orbit.

Let $\epsilon = \frac{N}{2}$, we can translate the periodic orbit an ϵ-distance in the perpendicular direction without intersecting another vertex anywhere in the expression, that gives an ϵ-neighborhood which have a periodic orbit, with the same initial direction as the original orbit. This neighborhood of initial positions and the corresponding periodic orbit with the same initial direction as the original orbit. These orbits form a strip of periodic orbit. These orbits are periodic since they all keep their direction in the correct orientation, they are periodic, and if the initial position is translated an amount in direction, then if a given point is translated in the same direction with the same amount, the translated point will be identified as well, which means that there exist a strip of periodic orbit is a part of .

Theorem 2.3.6. *[11] A rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{3}$ can be expressed as a regular hexagon with unfolding*

Theorem 2.3.7. *[11] In the class of a billiard tables that tile the plane with reflections is* {*the regular hexagons; the rectangle; the rhombus with angles* $2\pi/3$ *and* $\pi/6$ *;any kite with angles* $2\pi/3, \pi/2, \pi/2, and \pi/3$*; any triangle with angles* $2\pi/3$ *and two angles of* $\pi/6$ *; any triangle with one right angle, one angle of* $\pi/3$ *and one angle of* $\pi/6$*; any triangle with a right angle and two angles of* $\pi/4$ *; and any triangle with all angles* $\pi/3$ } *every table admits perpendicular periodic orbit.*

Proof:

- Regular hexagon, or any rectangle, for any two parallel edges of the polygon we can create an orbit that is perpendicular to these edges in which the orbit does not hit a vertex .

- In the case of rhombus with angles $\frac{2\pi}{3}$ and $\frac{\pi}{3}$ a regular hexagon can be obtained from the rhombus by reflection. Because a regular hexagon has a perpendicular periodic orbit. By creating an orbit which is perpendicular to every two parallel edges of the resulting regular hexagon from the rhombus. That does not strike any vertex.

- A $(\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6})$ triangle, begin with the orbit which strikes the largest edge of the obtuse triangle with an angle of incidence equals $\frac{\pi}{3}$, and since the two smallest angles of the obtuse triangle equals $\frac{\pi}{6}$ when the orbit strikes the shorter sides they must be right angles since the sum of degree of a triangle equal π. This orbit is periodic since it will end up in the same initial position with the same initial direction, since it has perpendicular angle of incidence and will not strike a vertex. Hence it has a perpendicular periodic orbit.

- The kite with angles $\frac{2\pi}{3}$,$\frac{\pi}{2}$,$\frac{\pi}{2}$ and $\frac{\pi}{3}$. A regular hexagon can be obtained by unfolding. Because the regular hexagon has a perpendicular periodic orbit so does the kite.
- A triangle $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ A rhombus $(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ can be obtained by reflecting this triangle. Since the resulting rhombus has a perpendicular periodic orbit so does the triangle.
- A triangle$(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$, this triangle can be expressed as a square. But the square is a special case of a rectangle, and the rectangle has a perpendicular periodic orbit so does the $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ triangle.
- Any equilateral triangle, using the unfolding it can be expressed as a hexagon, and since the hexagon has a perpendicular periodic orbit, so does the equilateral triangle ■.

Theorem 2.3.8. *[11] The class of a billiard tables that tile the plane with reflections is {the regular hexagons; the rectangle; the rhombus with angles* $2\pi/3$ *and* $\pi/6$*; any kite with angles* $2\pi/3, \pi/2, \pi/2, and\pi/3$*; any triangle with angles* $2\pi/3$ *and two angles of* $\pi/6$*; any triangle with one right angle, one angle of* $\pi/3$ *and one angle of* $\pi/6$*; any triangle with a right angle and two angles of* $\pi/4$*; and any triangle with all angles* $\pi/3$ *} every table admits a periodic orbit*

2.4 Billiards in square and rectangle

In 2011, Brooks studies the possible periodic trajectories of a point mass in a rectangular billiard [8]. And the number of impacts if it exists. It has been shown that these periodic trajectories depend on the slope of the initial trajectories. In this section we recall these results.

Theorem 2.4.1. *[8] Suppose that we draw a line with an irrational slope. Then it will never cross two different horizontal or (vertical) edges at the same point.*

Proof: Assume not, if that happens, then the slope between those corresponding points would be a ratio of two integers. But we chose the slope to be irrational, so this cannot happen. Hence, if we hit the ball with any irrational slope, its trajectory on the table will be nonharmonic .

following the similar argument, if we hit the ball with any rational slope, its trajectory on the table will be periodic. So there exist a nonperiodic trajectory ■.

Corollary 2. *[11] Given any rectangle whose ratio of dimensions is rational there exist an initial position for an orbit such that given that the orbit has a rational slope, then the orbit is periodic.*

Proof: because the square is one of the class of polygons that tile the plane with reflections so it admits a periodic trajectory ■.

2.4.1 Two impact periodic trajectory

Define the trajectory of the point mass by the direction perpendicular to the side containing the initial point the point-mass travels a long the perpendicular line from the initial point to the midpoint on the opposite side, and return along the same path. It will always return a long the same trajectory because the trajectory line is perpendicular to the sides of impact.

2.4.2 Four impact periodic trajectory

A four impact trajectory can take the shape of a square or a rectangle inside the square billiard domain.

Lemma 2.4.2. *[4] If a point-mass impact a line segment $\overline{AB}$ at n from a point r, not on $\overline{AB}$, and the trajectory of the point -mass is a long $\overrightarrow{np}$, then $\overrightarrow{np} \cap AB = \{n\}$.*

Proof : Since r is not on AB, $m\measuredangle rnA = \varrho$, for some $\varrho > 0$, by the mirror law of elastic impacts we get that $m\measuredangle rnA = m\measuredangle xnB$ for any $x \in \overrightarrow{np}$ with $x \neq n$ which implies that the measure of angle $\measuredangle xnB = \varrho > 0$, therefore x does not belong to $\overrightarrow{AB}$.
Which means that for any x belongs to $\overrightarrow{np}$ with $x \neq n$, x does not belong ■.

Square billiards

Consider a unit square billiard ABCD with A at $(0, 1)$, B at $(1, 1)$, C at $(1, 0)$ and D at $(0, 0)$, with the trajectory of a point mass starting on $\overline{AD}$ at $(0, \varrho)$
.

Claim 2.4.1. *[4] If the trajectory of a point mass is periodic ending at* $(x_0, y_0 \mp \varrho)$ *equivalent to* $(0, \varrho)$ *then* $x_0 \in 2\mathbb{Z}$.

Proof: In order for the point mass travel from the point $(0, \varrho)$ and coming back to any point on the line segment $\overline{AD}$, including $(0, \varrho)$, it must leave $\overline{AD}$ and impact $\overline{BC}$. With the trajectory unfolded in the plane, in order for the trajectory of the point mass to travel from the point$(0, \varrho)$ to $(x_0, y_0 \pm \varrho)$ equivalent to $(0, \varrho)$, the trajectory must pass through the line containing $\overline{BC}$. The square is originally positioned with $\overline{BC}$ at $x = 1$. Since we are investigating x_0, we are only concerned with what happens with the square when it is reflected horizontally since vertical reflection do not alter x_0. When the square is horizontally reflected, first it is reflected about the line $x = 1$. places the side of the square equivalent to $\overline{AD}$ on the line $x = 2$. Because the unfolding process consists only reflections by successive horizontal and vertical integer lines. If we continue reflecting the square by each successive integer integer line, the side equivalent to $\overline{AD}$ will always be placed on an even integer line. Also, these reflection will always place the side equivalent to $\overline{BC}$ on an odd integer line. Therefore, if $(x_0, y_0 \pm \varrho)$ is equivalent (x_0, ϱ) so it is impossible for x_0 to be an odd number. Therefore $x_0 \in 2\mathbb{Z}$ ■.

Claim 2.4.2. *[4] If the trajectory of point-mass is periodic ending at a point* $(x_0, y_0 + \varrho)$ *equivalent to* $(0, \varrho)$ *then* $y_0 \in 2\mathbb{Z}$

Proof: If the trajectory of a point mass ends at $(x_0, y_0 \pm \varrho)$and this point is equivalent to $(0, \varrho)$, then it must be possible to reflect the point mass by integer lines until the reflection of $(0, \varrho)$ is $(x_0, y_0 + \varrho)$, by successive reflection through integer lines starting with y =1, then $y = 2$ and so on, because the point is always a distance of either ϱ or $1 - 1\varrho$ from the nearest integer line the distance between two successive reflections of $(0, \varrho)$ is either 2ϱ or $2 - 2\varrho$.

After the first reflection, the distance between the two successive reflections of $(0, \varrho)$ will be $2-2\varrho$ after the next reflection the distance between $(0, \varrho)$ and the new reflected point is $2 - 2\varrho + 2\varrho$, while we continue with this reflection, we continue to alternate adding $2 - 2\varrho$ and 2ϱ. If we reflect an even number of times our new y- coordinate will be $r(2 - 2\varrho) + r2\varrho + \varrho$, for some $r \in \mathbb{N}$, which means that if $(x_0, y_0 - \varrho)$ is equivalent to $(0, \varrho)$, then

$$y_0 + \varrho = r(2 - 2\varrho) + r2\varrho + \varrho$$

or

$$y_0 - \varrho = r(2 - 2\varrho) + (r - 1)2\varrho + \varrho$$

with $r \in \mathbb{N}$

$$y_0 - 2\varrho = r(2 - 2\varrho) + (r - 1)2\varrho + \varrho - \varrho$$

$$y_0 - 2\varrho = 2r - 2r\varrho + 2r\varrho - 2\varrho$$

$$y_0 = 2r$$

Hence, $y_0 \in 2\mathbb{Z}$.

Similarly if $(x_0, y_0 + \varrho)$is equivalent to $(0, \varrho)$, then

$$y_0 + \varrho = r(2\varrho) + r(2 - 2\varrho) + \varrho$$

$$y_0 = 2r\varrho + 2r - 2r\varrho$$

So $y_0 \in 2\mathbb{Z}$, so if $(x_0.y_0 \pm \varrho)$ is the terminating point of the trajectory and is equivalent to $(0, \varrho)$ then $y_0 \in 2\mathbb{Z}$.

To have $(x_0, y_0 \pm \varrho)$ and the point $(0, \varrho)$ being an equivalent points while reflecting by these specific and successive integer lines $x_0, y_0 \in 2\mathbb{Z}$. This coincides with the method for unfolding trajectories by reflecting the billiard domain $\blacksquare$.

Claim 2.4.3. *[4] If the trajectory of a point- mass which starts on* $\overline{AD}$ *of* $(0, \varrho)$ *ends at a point* $(x_0, y_0 - \varrho)$ *with* $x_0, y_0 \in 2\mathbb{Z}$ *then the point terminates in a corner before the trajectory completes a period.*

Proof: The initial point of the trajectory is $(0, \varrho)$ and the terminating point is $(x_0, y_0 - \varrho)$. Therefore, the trajectory line is the line through these two points which means the midpoint of this line is $\frac{x_0}{2}, \frac{y_0}{2}$ and since $x_0, y_0 \in 2\mathbb{Z}$, which means that this point is a corner. Therefore the point mass impacts a corner before a period can be completed $\blacksquare$.

Claim 2.4.4. *[4] The trajectory of the point mass can not be periodic if the number of impacts, N, is odd*

Proof: As mentioned above $x_0, y_0 \in 2\mathbb{Z}$. In order for $x_0 \in 2\mathbb{Z}$, the number of horizontal reflections of the billiard boundary must be odd. And in order for $y_0 \in 2\mathbb{Z}$, the number of vertical reflections of the billiard boundary must be even initial point is on the integer line $x = 0$, but falls between integer lines vertically since the sum of an even number and odd number is an odd number, the total number of reflection to map the trajectory is odd. However the first impact occurs before we reflect the billiard domain, so we must add the first impact to this odd number of impacts so the total number of impacts is even. When $x_0, y_0 \in 2\mathbb{Z}$, this will always be the case, which we require for periodic trajectories, therefore if the number of impacts N in a trajectory is odd, the trajectory can not be periodic $\blacksquare$.

Lemma 2.4.3. *[4] Reflections about lines $x = \nu$ and $y = \omega$ with $\nu, \omega \in \mathbb{Z}$ preserve the parity of the coordinates of all integer points*

Proof: Let (c, d) be any point, with $c, d \in \mathbb{Z}$ reflect the point (c, d) by an integer line $y = \omega$, $\omega \in \mathbb{Z}$. Denote this reflection by $\sigma_{y=\omega}(c, d)$ by properties of reflection, we get that

$$\sigma_{y=\omega}(c, d) = (c, d + 2(\omega - d)) = (c, 2\omega - d)$$

because $\omega \in \mathbb{Z}$, 2ω is an even integer which means that if d is even $2\omega - d$ will be odd . Similarly, ω can reflect out point (c, d) by a vertical integer line, $x = \nu$, $\nu \in \mathbb{Z}$

$$\begin{aligned} \sigma_{x=\nu}(c, d) &= (c + 2(\nu - c), d) \\ &= (2\nu - c, d) \end{aligned}$$

will also be even $\blacksquare$.

Lemma 2.4.4. *[4] No composition of reflections about integer lines will ever map one side of our billiard boundary to the opposite side of the billiard boundary, so if the points $(x_0, y_0 + \varrho)$ and $(0, \varrho)$ are equivalent, $x_0, y_0 \in 2\mathbb{N}$.*

2.5 Billiards in an acute triangle

In this section we introduce the proof of Fagnano orbit in an acute triangle by dropping the altitudes from the vertices and drawing the orthic triangle the orbit following this triangle is what we call the Fagnano orbit [11].

Theorem 2.5.1. *[11] In an acute triangle there exists at least one non-perpendicular periodic orbit called the fagnano orbit.*

Proof: Suppose that $\triangle ABC$ is an arbitrary acute triangle, draw the altitudes from A,B, and C to points A_1, A_2, and A_3 respectively. The altitudes meet at a point called orthocenter O. $\angle OA_3B$ and $\angle OA_2C$ are right angles, the angles $\angle A_3OB$ and $\angle A_2OC$ are congruent vertical angles. Since the sum of the angles of a triangle is π, so $\angle A_3BO$ and A_2CO must be the same.

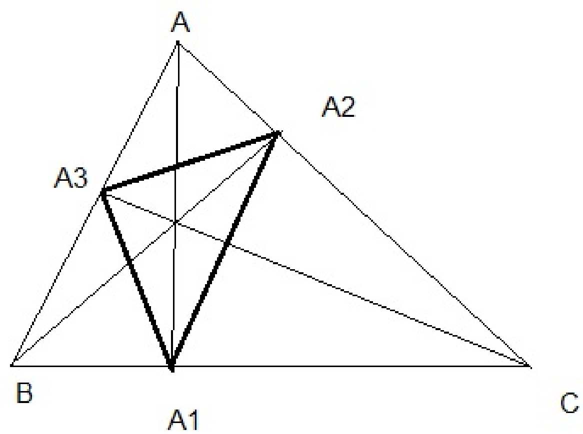

Figure 2.15: A fagnano orbit in an acute triangle

Circumscribe circle around the triangle $\triangle OA_1C$, we can find $\overline{OC}$ from the extended law of sines with respect to the circle about the triangle $\triangle OA_1C$. Let L_1 be the radius of the circle about $\triangle OA_1C$, then

$$\frac{\bar{OC}}{sin(\angle OA_1C)} = 2L_1.$$

But, $\angle OA_2C$ equals $90°$, so

$$\frac{OC}{sin(\angle OA_1C)} = \frac{\overline{OC}}{1} = 2L_1.$$

Then

$$\overline{OC} = 2L_1.$$

Using the same procedure for a circle around $\triangle OA_2C$. Since $\angle OA_2C$ is right and let L_2 be the radius of the circle circumscribed around $\triangle\triangle OA_2C$, then

$$\frac{\overline{OC}}{sin(\angle OA_2C)} = \frac{\overline{OC}}{sin(90^\circ)} = 2L_2.$$

$L_1 = L_2.$

If a circle around the point C, with radius L_1 and a circle around the point O with radius L_2 are drawn and meet at a point called H, then H is the center of both of the circles circumscribed around triangles $\triangle OA_1C$ and $\triangle OA_2C$.

Because both of the circles circumscribed around $\triangle OA_1C$ and $\triangle OA_2C$ have the same radius, they must be the same circle.

If two right angles share the hypotenuse, the quadrilateral formed by the composition of these two triangles, can be inscribed in a circle (since the sum of the opposite angles of the quadrilateral is $90^\circ + 90^\circ = 180^\circ$. If the circle circumscribes the quadrilateral CA_2OA, the angles $\angle A_2CO$ and $\angle OA_1A_2$ are equal, since both angles share the same arc-length in the circle.

By the same way the triangle $\triangle OA_3B$ and $\triangle OA_1B$ are right triangles that share the hypotenuse, As consequence $\angle A_1BO$ and $\angle OA_1A_3$ since they share the same arc-length. So,

$$\angle OA_1A_3 \cong \angle A_3BO \cong \angle A_2CO \cong \angle OA_1A_2.$$

Because the angles $\angle OA_1C$ and $\angle OA_1B$ are both equal 90°, and since the angles $\angle OA_1A_3$,$\angle OA_1A_2$ are congruent, so $\angle A_3A_1B$ and A_2A_1C are congruent, so we conclude that the angles $\angle A_1A_2C$ and A_3AA_2 and $\angle A_1A_3B$ and $\angle AA_3A_2$ are congruent.

The inscribed triangle $\triangle A_1A_2A_3$ is periodic orbit, since it has been proved that the incident angles are congruent to their corresponding angle of reflection.

The angle of incidence can not be right angle; since the triangle $\triangle ABC$ is acute and therefore the altitudes intersect at a point between the other two vertices of triangle $\triangle ABC$ which is not a vertex.

If the angle of incidence is a right angle, then the area of the inscribed triangle $\triangle A_1A_2A_3$ would be zero, that means this orbit is non perpendicular.

Let x_0 be any point on the boundary of the triangle $\triangle A_1A_2A_3$ to be an initial point of our orbit. If we take the direction of the boundary of the triangle $\triangle A_1A_2A_3$, this orbit will end up in the same initial positions, with the same initial direction. So it is periodic $\blacksquare$.

2.6 Billiards in right triangle

Periodic billiard orbits always comes in strips, that given any point $x = (z, v)$ where z belongs to the boundary of Z and v indicates the inward pointing direction, and x is a point whose billiard orbit is periodic, let $I \subseteq \partial Z$ be an open interval such that $z \in$ I and for any $z^{'} \in I$ the orbit of $x^{'} = (z^{'}, v)$ visits the same sequence of sides as x so the point $x^{'}$ is periodic. In this section we recall some of the results about billiards in right triangle found in [9],[22],[23] and [14].

Definition 2.6.1. *[22] A maximal width strip will be called* ***a beam.***

Theorem 2.6.1. *[9] Through every point of a right triangle passes a periodic trajectory.*

Construction of a periodic trajectories in right triangles

- Obtain a rhombus via reflecting the triangle in its legs, each diagonal of the rhombus is composed of two legs of the triangle. Call the vertices corresponding to the acute angles of the constructed rhombus the extremities[9].

- Reflect the rhombus in its sides by following the unfolding of a periodic trajectory or by constructing a corridor of rhombuses such that the first and last rhombuses are parallel to each other. The corresponding points in these rhombuses are joined by a segment, when this segment entirely contained in the corridor, it is the unfolding of a periodic trajectory[9].

The trajectories which ends at a vertex of an angle are called singular trajectories, and it is called a removable singularity if the angle is a right angle and having the form $\frac{\pi}{n}$[9].

Denote the smaller acute angle of a right triangle by α. So the acute angle of the rhombus is 2α. Because the rhombus is symmetric its reflection in a side is equivalent to a rotation around its extremity by an angle 2α or -2α, depending on the direction of rotation [9].

To prove theorem 2.6.1. First reflect the initial rhombus in its sides n times so as to make it turn counterclockwise by $2\alpha.n$ around the vertex B; then n more times so that it turns clockwise, by $-2\alpha.n$, around the vertex C. The number n change from 1 to $\lceil \frac{\pi}{2\alpha} \rceil$. The corridor of rhombuses has a center of symmetry it coincides with the center O_n of the nth rhombus.

Trace the maximality wide bundle of segments parallel to the line $O_0O_nO_{2n}$ and entirely contained in the rhombuses constructed, we call this bundle a strip. All segments of the strip (except its border) are unfoldings of periodic trajectories. These trajectories are called S-trajectories [9]. The maximum wide bundle of segments parallel to the line $O_0O_nO_{2n}$ and entirely contained in the rhombuses constructed. All the segments of the strips (except its borders) are unfolding of periodic trajectories in $\triangle BO_0A_1$. Since the corridor resembles the letter S these trajectories are called S trajectories.

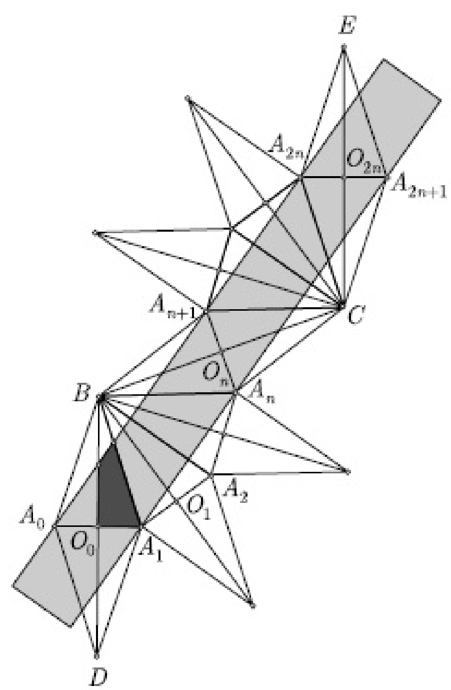

Figure 2.16: The strip corresponding to the maximal value n =N-1.

Prposition 5. *[9] S trajectories are perpendicular.*

Proof: The trajectories of the n-th strip are parallel to O_0O_n. So they meet either the longer leg BA_k or the hypotenuse BO_k of the triangle being perpendicular to it.

Prposition 6. *[9] The n-th strip is indicated to an angle $n\alpha$ with respect to the line O_0A_1.*

Proof : Case 1 : n is even
For every n, n = 2k, the sides of the angle $\angle A_1O_0O_n$ are perpendicular to the sides of the angles $\angle O_0BO_k = n\alpha$.

Case 2: n is odd
For an odd n, n = 2k-1, the $\angle A_0O_0O_n$ are perpendicular to the sides of the angle $\angle O_0BA_k$, and also we have $\angle A_1O_0O_n = \angle O_0BO_k = n\alpha$ ■.

Theorem 2.6.2. *[23] Periodic orbits are dense in the phase space of any irrational right triangle.*

Theorem 2.6.3. *[22] Suppose that $\mathbb{V}$ is an irrational right triangle. Then there exists an at most countable set $B \subset \mathbb{V}$ such that for every $\vartheta \in \mathbb{V}\backslash B$ the orbit (ϑ,θ) is periodic for a dense subset of directions $\theta \in \mathbb{S}^1$.*

This theorem tells us that except for an at most a countable set B of initial positions $0 \leq x_1 \leq x_2 \leq 1$ if (x_1, x_2) does not belong to B then the orbit of $(x_1, v_1), (x_2, v_2)$ is periodic for a dense set of velocities (v_1, v_2).

Suppose that one leg of the right triangle is horizontal with the smaller angle α with the smaller angle of the right triangle and its the angle between this leg and the hypotenuse and when we say perpendicular (orbit ,beam,..) belongs to the perpendicularity to the horizontal leg. when we work with billiards in a right triangle we reflect the triangle in the sides of right angle to obtain a rhombus so the study of billiards in this triangle reduces to the rhombus.

Theorem 2.6.4. *[22] For any irrational right triangle whose smaller angle satisfies $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$ consider any perpendicular periodic beam of period 2p. Then*

1. *the midpoint of the beam hits the right angle vertex of the triangle.*

2. *the beam returns to itself after half its period with the opposite orientation ,and*

3. *p is even and the first p+1 letters of the code of the beam from a palindrome.*

The proof based on the idea of unfolding that is instead of reflecting the trajectory reflect the beam itself in this side. Fix an orbit sequence so there is a sequence around this trajectory have the same which have the same sequence of reflections is made by all these trajectories in the strip, denote the number of reflections by the length of the strip. The boundary of a periodic beam contains at least one trajectory segments which hit a vertex of a polygon. If this vertex is the right angle, then the sequence of reflection on both sides of this vertex is essentially the same because the singularity due to such a vertex is removable.

Lemma 2.6.5. *[22] Suppose that A^+ and B^+ are parallel beams not necessarily (perpendicular or periodic) with the same codes, then $A^+ = B^+$.*

Proof: By induction, let n be the length of reflection.

For n=1. there is only one beam with code 0. Moreover there is a unique beam with code 01 and a single beam with code 01^{-1}. And they are separated by an orbit which passes through a vertex of the rhombus.

Suppose the assumption is true for any beam of length n. And it is the only beam with its code. Fix a beam of length n and let k be the last rhombus this beam visits. This rhombus is related to the $(k+1)st$ by two parallel sides, and to the $(k-1)st$ along the other two parallel sides. Two points are common to these three rhombi. This beam can hit exactly one of these two points when it passes through k-th rhombus. One when exiting and the other when entering the $k-th$ rhombus.

If this does not happen, then the beam does not split and it continued to a unique $n+1$ beam. If that happens, then the beam splits into two subbeams with $(n+1)$ length but with different codes (differing in the $(n+1)st$ place)[22] ■.

Theorem 2.6.6. *[22] Periodic orbit are dense in the phase space of irrational right triangles whose smaller angle satisfies* $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$.

The orbit is called recurrent if its code$\{a_i\}$ satisfies $a_j = a_0$ for some $j > 0$. A nonrecurrent orbit is called a positive escape orbit if $limsup_{i\to\infty} a_i = \infty$, and if $liminf_{i\to\infty} a_i = -\infty$.

A direction is called simple if there are no generalized diagonals in the invariant surface which contains the direction.

Theorem 2.6.7. *[22]*

- *Consider an irrational right triangle. In a simple direction then there is at most one nonsingular positive escape orbit and at most one nonsingular negative escape orbit .*
- *For any irrational right triangle whose smaller angle satisfies* $\frac{\pi}{6} < \alpha < \frac{\pi}{4}$ *there is at most one nonsingular escape orbit in the perpendicular direction. If it exists it is both a positive and a negative escape orbit.*

Theorem 2.6.8. *[19] Given a billiard table that is any arbitrary right triangle, there exists a perpendicular periodic orbit.*

Proof: Let $\triangle ABC$ be an arbitrary right triangle with the right angle at$\angle ABC$.

- When $\angle BAC = \angle ACB$ there exist perpendicular periodic orbit according to theorem (2.3.7).
- without loss of generality assume $\angle BAC < \angle ACB$ to represent all other cases. Use the unfolding across the line segment $\overline{AB}$ then we obtain a new triangle $\triangle ABC'$ then the larger triangle $\triangle ACC'$ is acute because $\angle BAC < \angle ACB$ then $\angle BAC < \frac{\pi}{4}$ so the larger angle of $\angle\angle CAC' = 2\angle BAC < \frac{\pi}{2}$ and both angles $\angle ACC' \cong AC'C < \frac{\pi}{2}$. Considering the large triangle $\triangle ACC'$. Because the large triangle $\triangle ACC'$ is acute so it has a fagnano periodic orbit with initial position at B. The other points where the fagnano orbit strikes the boundaries AC and AC' are D and E. When we shift our initial position B by ϵ another orbit. Label this new initial positions by F and the subsequent points where the orbit strikes the boundaries of the large triangle to be G, H, I, J and K.

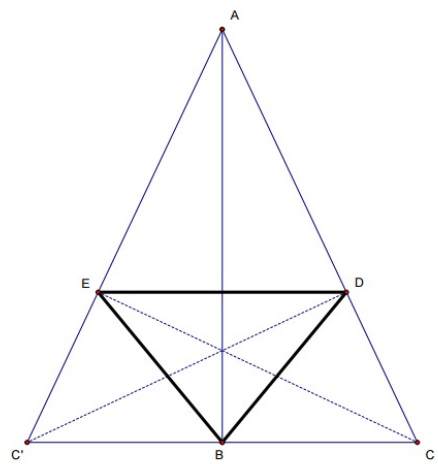

Figure 2.17: A fagnano orbit obtained by reflection a right triangle.

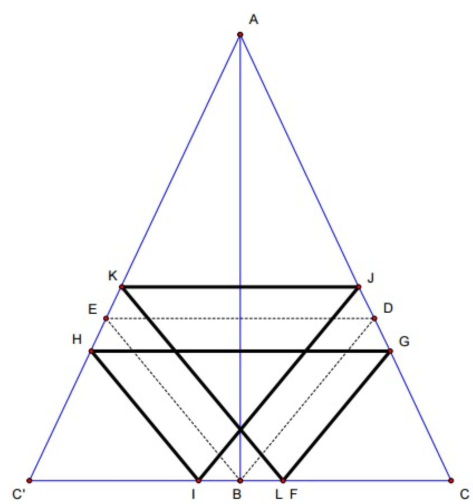

Figure 2.18: A Fagnano orbit after shifted by ϵ.

Since this orbit has the same initial direction $\bar{BD}\|\bar{FG}$ and so $\angle BDC \cong \angle FGC$. Since we are looking at orbits $\angle ADE \cong \angle BDC$. Because we are looking at orbits

$$\angle ADE \cong \angle BDC \cong FGC \cong AGH.$$

Hence $\bar{DE}\|\bar{GH}$. Repeating the same procedure for each set of angles shows

$$\angle ADE \cong \angle BDC \cong \angle FGC \cong \angle AGH \cong \angle AJK \cong \angle CJI$$

$$\angle AKJ \cong \angle AED \cong \angle AHG \cong \angle C^{'}HI \cong \angle C^{'}EB \cong \angle C^{'}KL$$

$$\angle C^{'}IH \cong \angle C^{'}BE \cong \angle C^{'}LK \cong \angle CFG \cong \angle CBD \cong \angle CIJ$$

Since the triangle $\triangle AC^{'}C$ is isosceles so the angle $\angle AC^{'}C$ is the same as the angle $\angle ACC^{'}$and since $\angle CFG \cong \angle C^{'}IH$ and because the sum of the angles in every triangle is equal to π we have $\angle IHC^{'} \cong \angle FGC$. Combining the two congruences lines, we have
$\angle ADE \cong \angle BDC \cong \angle FGC \cong \angle AGH \cong \angle AJK \cong \angle CJI \angle AKJ \cong \angle AED \cong \angle AHG \cong \angle C^{'}HI \cong \angle C^{'}EB \cong \angle C^{'}KL$.

We conclude that the triangles $\triangle AJK, \triangle ADE$and $\triangle AGH$ are all isosceles and similar to triangles $A\bar{C}C'$ and $\bar{DE}\|\bar{KJ}\|\bar{GH}\|\bar{CC'}$.

Since the isosceles triangles $\triangle AGH$ $\triangle ACC^{'}$, $\triangle AKJ$ share the angle $C^{'}AC$ and their bases are parallel, then the segments $\bar{GC} \cong \bar{HC'}$ and $\bar{KH} \cong \bar{JG}$. So $\triangle FGC \cong \triangle IHC^{'}$. Since they have a congruent side adjacent to two congruence angles.

$\overline{BC} \cong \overline{BC'}$ because $\overline{BC'}$ is a reflected copy of $\overline{BC}$. $\overline{BC} = \overline{FC} + \epsilon, \overline{BC'} = \overline{C'I} + \overline{IB}$, and $\overline{FC} \cong \overline{IB}$ because the triangle FGC is congruent to the triangle $C^{'}HI$ and the corresponding sides are also congruent, then $\overline{IB} = \epsilon$.

The triangles $\triangle FGC$ and $\triangle IJC$ are similar so:

$$\frac{\overline{FC}+2\epsilon}{\overline{FC}}=\frac{\overline{CG}+\overline{JG}}{\overline{CG}}$$

$$(\overline{FC}+2\epsilon)\times\overline{CG}=(\overline{CG}+\overline{JG})\times\overline{FC}$$

$$\overline{FC}\times\overline{CG}+2\epsilon\overline{CG}=\overline{CG}\times\overline{FC}+\overline{JG}\times\overline{FC}.$$

Delete $\overline{FC}\times\overline{CG}$ from both sides then we have

$$2\epsilon\overline{CG}=\overline{JG}\times\overline{FC}$$

$$\frac{\overline{CG}(2\epsilon)}{\overline{FC}}=\overline{JG}.$$

The triangle $\triangle C'HI\cong C'Kl$, the ratio of their sides gives:

$$\frac{\overline{KH}+\overline{HC'}}{HC'}=\frac{\overline{C'I}+\epsilon+\overline{BL}}{\overline{C'I}}$$

$$\overline{C'I}\times\overline{KH}+\overline{HC'}\times\overline{C'I}=\overline{HC'}\times\overline{C'I}+\overline{HC'}\times\epsilon+\overline{HC'}\times\overline{BL}$$

Delete $\overline{HC'}\times\overline{C'I}$ from both sides we get

$$\overline{C'I}\times\overline{KH}=\overline{HC'}\times\epsilon+\overline{HC'}\times\overline{BL}$$

$$\overline{C'I}\times\overline{KH}=\overline{HC'}(\epsilon+\overline{BL})$$

$$\frac{\overline{KH}}{\overline{HC'}}=\frac{\epsilon+\overline{BL}}{\overline{C'I}}.$$

But, $IB=\epsilon$, $\overline{KH}=\overline{JG}$, $\overline{GC}\cong\overline{HC'}$ and $\overline{FC}\cong\overline{C'I}$ so we obtain

$$\frac{\overline{JG}}{\overline{GC}}=\frac{\epsilon+\overline{BL}}{\overline{FC}}$$

$$\frac{\frac{\overline{GC}(2\epsilon)}{\overline{FC}}}{\overline{GC}}=\frac{\epsilon+\overline{BL}}{\overline{FC}}$$

$$\overline{BL}+\epsilon=2\epsilon$$

$$\overline{BL}=\epsilon$$

So if $\epsilon = \overline{BL}$ and $\epsilon = \overline{BF}$ then $\overline{BL} \cong \overline{BF}$ and so L is F which means that the orbit return to its initial position and it will have the same initial direction because $\angle C'LK \cong \angle CFG$. So it is a periodic orbit. Since $\overline{AB}$ is perpendicular to $\overline{CC'}$and $\overline{GH} \| \overline{KJ} \| \overline{CC'}$ and $\overline{AB}$ is perpendicular to CC' then it is perpendicular to $\overline{GH}$ and $\overline{KJ}$ as well $\blacksquare$.

2.7 Periodic orbits of Billiards on an equilateral triangle

In this section we give some basic results about billiards on an equilateral triangle proved in [1] and [2].

2.7.1 Fagnano's Orbit

Theorem 2.7.1. *[1] An equilateral triangles admits a periodic orbit*

Proof:

1. Case 1: Start at the midpoint with an angle of incidence equals 60°, and the bouncing points at the midpoint of each side, then the resulting is the orbit of period three.

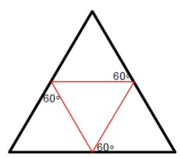

Figure 2.19: An orbit of period 3 in an equilateral triangle.

2. Case 2: The starting point is no at the midpoint of the edge, then the orbit of period 6 appears.

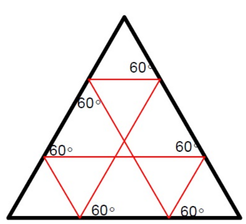

Figure 2.20: An orbit of period 6 in an equilateral triangle.

■.

2.7.2 Orbits and tesselations

Let $\triangle ABC$ be an equilateral triangle

Prposition 7. *[2] Every nonsingular trajectory strikes some side of* ΔABC *with an angle of incidence in the range* $30° < \theta < 60°$.

Proof: Let γ be a nonsingular trajectory, let x_1 be a point where γ hits $\triangle ABC$ with angle of incidence β_1.

- If β_1 lies in the interval $30° < \beta_1 < 60°$; set $\theta = \beta_1$.
- Else, let γ_1 be the line segment joining. x_1 to the next incident point x_2. Let x_1 lies on a side $\overline{AC}$ and x_2 is on side $\overline{BC}$.

  1. If $0 < \beta_1 < 30°$, then $\beta_2 = m\angle x_1 x_2 B$. Since β_2 is the exterior angle of the triangle $\triangle x_1 x_2 C$ so $\theta_2 = 60° + \theta_1$. Hence $60° < \theta_2 < 90°$. Suppose that γ_2 is the segment of γ joining x_2 to the next incident point x_3. Then the angle of incidence at x_3 lies in the range $30° < \beta_3 < 60°$ since the triangle $\triangle x_2 x_3 C$ have angles $60°$ at C and $\angle x_3 x_2 C$ equals $60 + \beta_1$ because of the mirror law. So $\beta_3 = 180° - (60° + \beta_1 + 60°) = 60° - \beta_1$, because β_1 lies in the range $0 < \beta_1 < 30°$ so $30° < \beta_3 < 60°$. Set $\theta = \beta_3$.
  2. If $60° < \beta_1 < 90°$ and

     (a) If β_1 is an interior angle of $\triangle x_1 x_2 C$, then $\beta_2 = 180° - (60 + \theta_1) = 120 - \theta_1$. Because $60° < \beta_1 \leq 90°$, so the minimum value of β_2 is $120 - 90 = 30$ so the maximum value is $120 - 60 = 60$, so $30° < \beta_2 < 60°$. Set $\theta = \beta_2$.

     (b) If β_1 is an exterior of $\triangle x_1 x_2 C$. Then $\beta_1 = 60° + \beta_2$ and $\beta_2 = \beta_1 - 60°$. Since β_1 lies in the interval $60° < \beta_1 \leq 90°$, so $\beta_2 \in 0 < \beta_2 < 30°$. So β_3 is the exterior angle of $\triangle x_2 x_3 B$ and equals $\beta_2 + 60$. Let γ_3 be the segment of γ that connects x_3 to the next strike point x_4. So

     $$\beta_4 = 180° - (60° + \beta_3) = 120 - \beta_3 = 120 - (60 + \beta_2) = 60 - \beta_2.$$

     Hence, β_4 lies in the desired interval. Set $\theta = \beta_4$.

■.

Define the periodic orbit on the triangle $\triangle ABC$ by γ. This triangle is oriented so that its base is horizontal. Pick a point p at which γ strikes the triangle $\triangle ABC$ with angle of incidence in the range $30° \leq \theta \leq 60°$.

A regular tessellation of the plane by equilateral triangles each one is congruent to $\triangle ABC$ and positioned so that one of its families of parallel edges is horizontal will be dented by τ.

Embed $\triangle ABC$ in τ so that its base BC is collinear with a horizontal edge of τ. Let $\gamma_1, ..., \gamma_n$ denote the direct segments of γ labeled sequentially, then γ_1 begins at p and terminates at p_1 on side s_1 of $\triangle ABC$ with angle of incidence θ_1.

Prposition 8. *[2] A periodic orbit strikes the sides of $\triangle ABC$ with at most three incidence angles exactly one of which lies in the range $30° \leq \theta \leq 60°$. In fact, exactly one of the following holds:*

- *All incidence angles measure $60°$.*
- *There are exactly two distinct incidence angles measuring $30°$ and $90°$.*
- *There are exactly three distinct incidence angles ϕ, θ and ψ, $0 < \phi < 30° < \theta < 60° < \psi < 90°$.*

Proof : Define the periodic orbit by γ, and PQ to be an unfolding. By construction, PQ cross each horizontal edges of τ with angle of incidence lies in the range $30° \leq \theta \leq 60°$. As consequence, The unfolding cuts a left leaning edge of τ by an angle of incidence ϕ which equals $120 - \theta$, since $\theta + \phi + 60° = 180°$, and cross a right-leaning edge of the tessellation by a angle of incidence equals ψ. $\psi = 180 - (60 + \theta + 60) = 60 - \theta$.

- when $\theta = 60°$, $\psi = 0$, PQ does not cross a right leaning so it crosses the left-leaning and horizontal edges of τ. In this case all the incidence angles are equal measures $60°$. In this situation γ is either the fagnano orbit, and a primitive orbit of period 6, or some iterates of these.
- When $\theta = 30°$, then $\phi = 120° - 30° = 90°$, and $\psi = 60° - 30° = 30°$. In which case γ is either of period four or some iterate of it.

- When $30^\circ < \theta < 60^\circ$, so the lower limit value of ϕ is $120^\circ - 60^\circ = 60^\circ$, and the upper limit value of ϕ equals $120^\circ - 30^\circ = 90^\circ$. Therefore $60^\circ < \phi < 90^\circ$, the lower limit of ψ equals $60 - 60 = 0$ and the upper limit of $\psi = 60 - 30 = 30$, so ψ lies within the range $0^\circ < \psi < 30^\circ$.

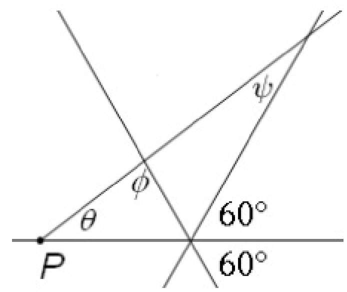

Figure 2.21: Incidence angles θ, ϕ, ψ.

Corollary 3. *[2] Any two unfoldings of a periodic orbit are parallel.*

Theorem 2.7.2. *[2] If an unfolding of a periodic orbit γ terminates on a horizontal edge of τ, then γ has even period.*

Proof: If γ is a periodic orbit with the unfolding PQ. P and Q exist on a horizontal edges of τ, and the basic triangles of τ. Cut by PQ pair off, and obtain a polygon of rhombic tiles [2]. When the path PQ passes through the resulting polygon, the path cuts one of the edges of each rhombic tile, crosses a diagonal of that tile (collinear with a left leaning edge of τ), and then exits the rhombic through a distinct edge. Because the exits edge of one tile is the entrance edge of the next one, and the edge containing P in the first rhombus of the tile is identified with the edge containing Q. Therefore the number of distinct edges PQ crosses equals twice the number of rhombic tiles. Therefore α has even period $\blacksquare$.

Theorem 2.7.3. *[2] If α is periodic orbit and $\gamma \neq \alpha^{2k-1}$ for all $k \geq 1$ then every unfolding of γ terminates on a horizontal edge of τ.*

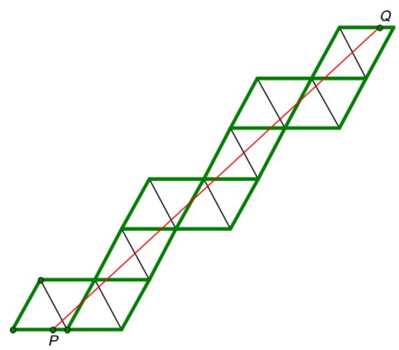

Figure 2.22: A typical rhombic tiling.

Proof: By contrapositive. Let PQ be an unfolding of a periodic orbit γ. And suppose β to be the angle of incidence at Q. Which is also the angle of incidence at P, and β belongs to $\{30°, 60°\}$, by the proof of proposition 8.

When $\beta = 30°$ then γ is some iterate of period four whose unfoldings terminate on a horizontal edge of τ. Therefore $\beta = 60°$. But, γ can not be an iterate of period six, and can not be an even iterate of γ. Because of that happens, then their unfoldings will terminate on a horizontal edge of τ so γ can not be an even iterate so $\gamma = \alpha^{2k-1}$for some $k \geq 1$ ■.

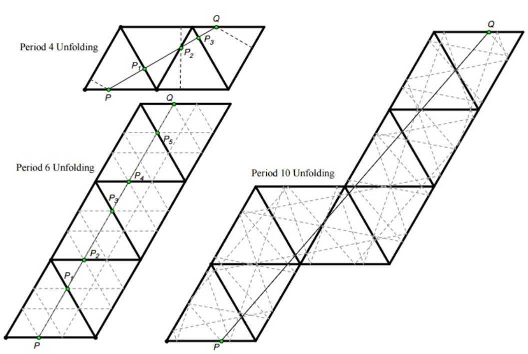

Figure 2.23: Unfolding orbits of period 4,6 and 10.

Corollary 4. *[2] If α is an orbit with odd period, then $\alpha = \gamma^{2k-1}$for some $k \geq 1$, in which case the period is $6k - 3$*

Suppose that γ is an even periodic orbit and let PQ be an unfolding, and suppose that G is the group generated by all reflections in the edges of τ. Since the action of G on $\bar{BC}$ generates a regular tessellation $\mathcal{H}$ of the plane by Hexagons, γ ends on some horizontal edge of τ. As in the definition of an unfolding , suppose $\sigma_1, ..., \sigma_{n-1}$ be the reflections in the lines of τ cut by $\underline{PQ}$ in order and σ_n be the reflection in the line of τ containing Q. So the composition $f = \sigma_n \sigma_{n-1} ..... \sigma_1$ maps P to Q and maps the hexagon whose base $\bar{BC}$ contains P to the hexagon whose base $\overline{B'C'}$ and contains the point Q. Then the period of γ (n) is even and f is either translation by vector $\vec{PQ}$ or rotation of 120° or 240°. But $\bar{BC} \| \bar{B'}C'$ so f is a translation and the position of Q on $B^{'}C'$ is exactly the same as the position p on BC is exactly the same as the position P on $\bar{BC}$.

Definition 2.7.1. *[2] Periodic orbits α and β are equivalent if there exist respective unfolding $\underline{PQ}$ and $\underline{RS}$ and horizontal translation τ such that $\underline{RS}$ $= \tau(PQ)$.*

- The symbol $[\alpha]$ denote the equivalence class of α .
- The period of class $[\alpha]$ is the period of its elements.

Definition 2.7.2. *[2] a class is even if and only if it has even period.*

Consider the unfolding $\underline{PQ}$ of a period orbit α. If $[\alpha\ ]$ is even, let R be a point on $\bar{BC}$, let τ be the translation from P to R . The point R is singular for $[\alpha]$, if $\tau(PQ)$ contains a vertex of τ; then $\tau(PQ)$ is an unfolding of a period orbit whenever R is nonsingular for $[\alpha]$. Also α strikes $\underline{BC}$ at finitely many points on $\bar{BC}$ and singular for $[\alpha]$. Therefore $[\alpha]$ has cardinality c (the cardinality of an interval) .
An odd period is γ^{2k-1} for some $k \geq 1$. But if $k \neq l$ then γ^{2k-1} and γ^{2l-1} have different period and cannot be equivalent. Therefore $[\gamma^{2k-1}]$ is a singleton class for each k .

Prposition 9. *[2] The cardinality of a class is determined by its parity, in fact α has odd period if and only if $[\alpha]$ is a singleton class*

Any two unfoldings whose terminal points lies on the same horizontal edge of $\mathcal{H}$ are equivalent. Since $\mathcal{H}$ has countably many horizontal edges, there are countably many even classes of orbits.
Furthermore, since at most finitely many points in $\bar{BC}$ are singular for each

even class .
There is a point 0 on $\bar{BC}$ other than the midpoint that is nonsingular for every class. Therefore given an even class $[\alpha]$, there is a point S and an element $x \in [\alpha]$ such that $\underline{OS}$ is an unfolding of α, then OS is the horizontal translation of PQ by $\vec{PO}$. Therefore α uniquely determines the point S denoted hence forth by S_α, OS_α denotes the fundamental unfolding of $[\alpha]$. The fundamental region at O, denoted by Γ_O, is the polar region $30° \leq \theta \leq 60°$ and centered at O. The S_α points are called the lattice point of Γ_O.

2.7.3 Orbits and rhombic coordinates

Let the vector $\vec{OS}$ denotes the fundamental unfolding$\underline{OS}$. Use the natural rhombic coordinate system given by τ. Suppose that O is the origin and let the horizontal line containing O be the x-axis and let the y-axis be the line passing through O with inclusion $60°$ and BC the unit in length.

$$\Gamma_O = \{(x, y) \mid 0 \leq x \leq y\}.$$

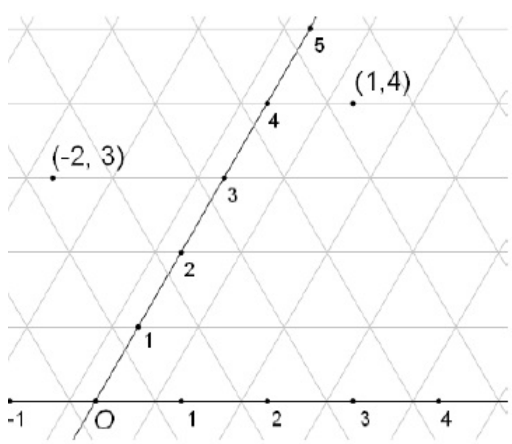

Figure 2.24: Rhombic coordinates.

Corollary 5. *[2] If $S_\alpha = (x, y)$ then α has period $2(x + y)$*

Definition 2.7.3. *[1] Let Γ be an orbit with initial point point P on edge e and terminal point on line segment l, then l lies in the bounce circle with* *radius n and center P, where $n \in \mathbb{N}$. If Γ crosses n line segment without the edge e, The edge e is the bounce circle of radius 0, and the bounce circle of radius ∞ is defined to be the line segment containing e, minus e itself.*

Theorem 2.7.4. *[1] The equilateral triangle admits exactly one periodic orbit of odd period, the period three orthoptic orbit.*

Proof: Define the vector representation of a periodic orbit by Λ. Which have E as its initial point and D to be its terminal point on a bounce circle of odd radius. Define the angle between Λ and the edge containing E by ϖ, and ξ to be the angle between Λ and the edge containing D. Because Λ is periodic, so $\varpi = \xi$.

Case 1 When D lies on a right -leaning diagonal, then $\varpi = \xi = 30°$, since both angles are equal and the third angle equals 120°. But when $\varpi = 30°$ does not yield any crossings with odd bounce circles on right - leaning diagonals, deleting this case.

Case 2 When D lies on a left leaning diagonal, then $\varpi = \xi = 60°$ then by 2.7.1, shows that $\varpi = 60°$ gives the orbit of period three or six ■.

Theorem 2.7.5. *[1] An orbit vector (u, v) is periodic if and only if x and y are integers such that $u \equiv v(mod 3)$.*

Proof: We say that a given vector (u, v) is periodic if and only if the point (u, v) after a finite number of reflection is the image of the origin and lies on a horizontal edge.

Highlighting all images of the edge contains the origin reveals a tessellation of the plane by hexagons.

The vectors $(-1, 2)$and $(1, 1)$ form a basis of the periodic orbits. Since both are images of the origin and lie on a horizontal edges so that it represents a periodic orbits. Any image of the origin which lies on a horizontal edge can be written as $a(-1, 2) + b(1, 1)$ for some a,b belong to integers. Because $-1 \equiv 2$ (mod 3) and $1 \equiv 1$ (mod 3), if $(x, y) = a(-1, 2) + b(1, 1)$ that is a periodic orbit then $u \equiv v$ mod 3 ■.

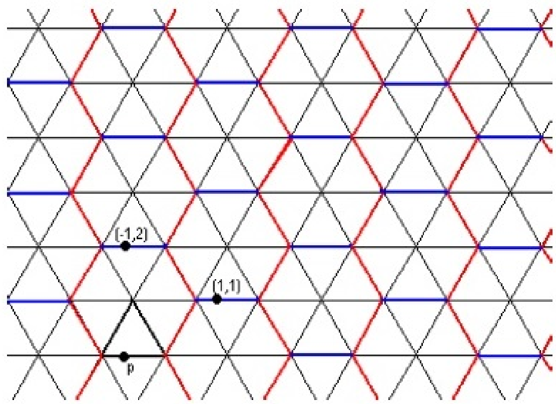

Figure 2.25: Periodic orbits initiate at P and terminate on a blue edge.

Theorem 2.7.6. *[1] The period of a periodic orbit* (x, y), *is given by*

$$\begin{cases} 2(x+y) & x \geq 0, y > 0 \\ 2y & x < 0, y > -x \\ -2x & x < 0, 0 < y \leq -x \end{cases}$$

Prposition 10. *[1] For any orbit not necessary periodic on the equilateral triangle, there are no more than three different bounce angles, with at least one between* $30°$ *and* $60°$, *inclusive .*

Proof: There are three sets of parallel lines, each line makes an angle of $60°$ with the other two lines. The orbit τ may be parallel to one of the three set of lines or intersects all sets of lines. The second case happens when $\alpha < 60°$. $\beta = 60° + \alpha$ since it is an exterior angle, and $\gamma = 60° - \alpha$ because the complement angle of γ is $120° + \alpha$. Therefore α or γ is the desired angle which lies in $[30°, 60°]$. In the case $\alpha = 60°$, the orbit τ is parallel to one of the diagonals and $\alpha \in [30°, 60°]$ ■.

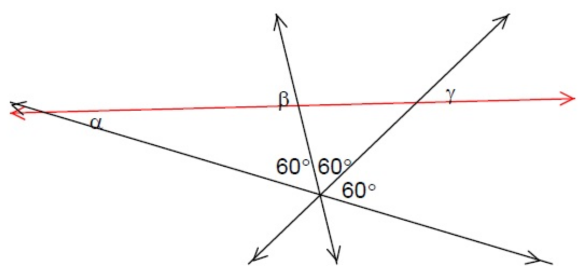

Figure 2.26: The orbit τ cuts the three parallel lines at angles α,β and γ

Chapter 3

Billiards on obtuse triangle

This chapter explains the MC-billiard programm wrote by Hooper and schwartz, much more details found about this program found in [20], [21], and [13].

Definition 3.0.1. *[20] Any periodic billiard path in T gives rise to an infinite repeating word that records the succession of sides encountered by the billiard path this periodic word is called the combinatorial type of the path.*

Theorem 3.0.1. *[20] Let T be an obtuse triangle whose big angle is at most 100 degrees then T has a stable periodic billiard path.*

Proof: Define the parameter space of obtuse triangles by Ω. The point (x, y) belongs to the parameter space Ω where x and y represent the small angles in radians of the triangle. Let W represents the combinatorial type which means the infinite periodic word which records the sequence of successive sides encountered by the billiard path [20]. To every W assign the region $O(W)$ containing the points (x, y) belongs to the parameter space such that W is a combinatorial type on the triangle whose smaller angle x and y. $O(W)$ is called an orbit tile if it is nonempty. The periodic billiard path corresponding to W is called stable if and only if the orbit tile $O(W)$is open [20] (i.e O(W) is a nonempty open set) and O(W) is contained in a line if it is not an open set.

Define S_{100} to be a region of the parameter space associated with triangles with the largest angle not exceeding one hundred degree assuming $x \leq y$ in S_{100}. Let $p_k = (\frac{\pi}{k}, \frac{\pi}{2} - \frac{\pi}{k}) \in \partial\Delta$.
p_k corresponds to a right triangle .

3.1 Unfoldings

We will restrict the work here to words of even length. The part of an infinite periodic word is called the finite word [20]. We can also get the infinite periodic word from the given finite word via repeating it endlessly.

Let a word $W = w_1, ..., w_{2k}$ be given. If we reflect the given triangle we will obtain a sequence of triangles $T_1, ..., T_{2k}$ such that the triangle T_{i-1} is related to the side T_i via reflection across the w_ith edge of T_i, $i = 2, ..., 2k$. Denote the unfolding corresponding to the pair(W,T) by $U(W,T)$ such that $U(W,T) = \{T_i\}_{i=1}^{2k}$. Label the top vertices of $U(W,T)$ as $a_1, a_2, ...$ from left to right. and by the same way let $b_1, b_2, ...$ be the bottom vertices of the unfolding.

If the following hold then we say the word W is stable

1. The first and last edge of U(W,T) are parallel.
2. There is a line segment L joining equivalent interior points on the first and last edges that remains entirely inside $U(W,T)$ which is called the centerline.

and the converse is also true [20].

3.1.1 Stability

Definition 3.1.1. *[20] Aperiodic billiard path on a triangle is stable if nearby triangles have a periodic billiard path of the same combinatorial type. We say that a combinatorial type W is stable if the first and last sides of $U(W,T)$ are parallel for any triangle T.*

Lemma 3.1.1. *[20] If W is the combinatorial type for a stable billiard path, then W is a stable word. Conversely, if W is a stable word, then any periodic billiard path described by W is stable.*

Proof: Let $U(W,T)$ denotes the unfolding corresponding to a stable periodic billiard path W on T, then the first and last side of $U(W,T)$ are parallel and $U(W,T)$ has a centerline. If we perturb a triangle T slightly we will obtain a new triangle T', according to the definition of the stability this new triangle still have a periodic billiard path with the same combinatorics. Therefore the new unfolding $U(W,T^{'})$ still has a center line and the first and last side of the new unfolding are still parallel. It has been shown that the first and last side of $U(W,*)$ are still parallel for an open set of triangles.

Conversely, let the stable word W denotes a periodic billiard path on T. If we perturb the corresponding triangle slightly, then the first and last sides of the new unfolding are still parallel. In particular all the a-vertices still lie above all the b-vertices. Which means the nearby triangles still have a periodic billiard path described by W■.

Definition 3.1.2. *[20] Given a word W then N_{ij} denotes the number of couplets having type ij.*

Example 5. *Let W = 13 31 21 12 23 23 13 21*

In this example we have $N_{12} = 1$,$N_{21} = 2$, $N_{13} = 2$, $N_{31} = 1$,$N_{23} = 2$ and $N_{32} = 0$.

Lemma 3.1.2. *[20] A word W is stable if and only if*

$$N_{12} - N_{21} = N_{23} - N_{32} = N_{31} - N_{13}$$

Proof: Let U(W,T) be the unfolding corresponding to the word W and let $T_1, ..., T_{2k}$ be the triangles in this unfolding. At the beginning of the unfolding U(W,T) add a triangle T_0 such that $T_0$and T_1 are related via reflection across the first edge. The first and last sides are parallel if and only if T_0 and T_{2k} are related via a translation [20].

Define α_i to be the angle opposite to the side i in the triangle. Consider the even triangles of the unfolding, N_{12} represents the number of times a triangle T_i is rotated in to the triangle T_{i+2} in the counterclockwise direction. Similarly for N_{21} but in the clockwise direction. The procedure is the same for N_{13},N_{31} and N_{32}, N_{32}. Therefore we rotate T_0 to get T_{2k} by an angle equals.

$$2N_{12}\alpha_3 - 2N_{21}\alpha_3 + 2N_{23}\alpha_1 - 2N_{32}\alpha_1 + 2N_{31}\alpha_2 - 2N_{13}\alpha_2$$

$$2(N_{12} - N_{21})\alpha_3 + 2(N_{23} - N_{32})\alpha_1 + 2(N_{31} - N_{13})\alpha_2 \tag{3.1}$$

The map taking T_0 to $T_{2k}$is a translation if and only if the equation(3.1) equals an integer multiple of 2π [20] ■.

The second way to describe stability

Given the word W then the hexpath$\mathcal{H}$ associated to this word can be drawn from the given word by moving a long the dth family when we encounter the digit d.

Lemma 3.1.3. *[20] A word W is stable if and only if the number of times each letter $l = 1, 2, 3$ appears in an odd position in W equals the number of times l appears in an even position.*

Example 6. *Let*

$$W = 12312, W^2 = 1231212312$$

the digit 1 appears in positions 1,4,6,9 it appears twice in even positions and twice in odd positions, for $l = 2$ appears twice in even positions 2,10 *,and twice in odd positions* 1 *and* 9*, similarly for $l = 3$ appears in odd position 3 and in even position 8.*

Corollary 6. *[20] If W is a word of odd length, then W^2 is stable*

Proof: Let W be a word of odd length then $W = w_1w_2.....w_{2k+1}$ then $W^2 = w_1w_2....w_{2k+1}w_1w_2....w_{2k+1}$, let $l \in \{1, , 3\}$ appears n times in odd positions and m times in even position then in W^2 l appears $n + m$ times in even positions and $n + m$ times in odd position, so according to lemma 3.1.3 W^2 is stable.

Remark 3. *[20] The word window in McBilliard draws the hexpaths for the combinatorial types that the search engine finds.*

Lemma 3.1.4. *[20] Let $W = w_1, ..., w_{2n}$. Let n_{dj} denote the number of solutions to the equation $w_i = d$ with i congruent to j modulo 2. Let $n_d = n_{d0} - n_{d1}$. Then W is stable if and only if $n_d(W)$ is independent of d.*

Proof: Define the angles of the triangle T by $\alpha_1, \alpha_2, \alpha_3$. Going from T_0 to T_{2n}, the number of times a triangle is rotated about the jth vertex counter-clockwise is denoted by n_j. Each time we do such a rotation it is by $2\alpha_j$. Therefore to obtain $T_{2n}$from T_0, the angle through which we translate and rotate is $2n_1\alpha_1 + 2n_2\alpha_2 + 2n_3\alpha_3 = 2n_1(\alpha_1 + \alpha_2 + \alpha_3) = 2n_1\pi$ since the sum above is an integer multiple of 2π so the map take T_0 to T_{2n} is a translation. Hence these triangles are parallel■.

We can draw a useful graphical interpretation of the word W in a word window in MCBilliards. The planner hexagonal tiling $\mathcal{H}$ represent the union od edges each edge is labeled by either $1, 2$ or 3 depending on the family containing it. Let W be a word, a path P(W) in $\mathcal{H}$ can be constructed, in succession according to the digits of W.

Lemma 3.1.5. *[20] A word is stable if and only if its hexpath is closed.*

3.1.2 Special palindrome

Definition 3.1.3. *[20] W is called a special palindrome if W is stable and has the form $W = dCd(C^{-1})$ where C is a subword and C^{-1} is the reverse of C and $d \in \{1, 2, 3\}$. In this case $U(W, T)$ has bilateral symmetry and the translation carrying the first side to the last sides moves perpendicular to these sides.*

If there is a centerline in the unfolding U(W,T), then it must be perpendicular to the first and last sides. Therefore the associated periodic billiard path on T starts and ends perpendicular to one of the sides of T [20]. Conversely, a stable periodic billiard path in the triangle T satisfying the above property has a word W that is special palindrome.

3.1.3 Turning angles and turning pairs

Definition 3.1.4. *[20] Suppose that $U(W, t)$ is the unfolding and let e_1 be the first edge, oriented so that it points from b_1 to a_1. U(W,T) is said to be in the first position if e_1 is parallel to $(0, 1)$ i.e, e_1 points in the direction of the positive Y- axis.*

Let e be an oriented edge of U(W,T) and if we rotate the positive y-axis counterclockwise until it coincide with e denote the resulting angle by $\theta(e)$ according to the definition $\theta(e_1)$ must be equal to zero. In general $\theta(e)$ is defined modulo 2π [20].

Define $\theta(e)$ to be a function of $(x, y) \in \Omega$, and can be written as

$$\theta(e; x, y) = M(e)x + N(e)y + \epsilon\pi \quad mod \quad 2\pi \quad \epsilon \in \{0, 1\}. \tag{3.2}$$

Consider unoriented edges in which case we have

$$\theta(e; x, y) = M(e)x + N(e)y \quad mod \quad \pi \tag{3.3}$$

$(M(e), N(e))$ is the turning pair for e.

Construction of the angular correspondence: Define a canonical map that take the triangle T_i from the set of triangles of the unfolding to the *i-th* vertex v_i of the hexpath. The edge between T_i and T_{i+1} of the unfolding $U(T, *)$ corresponds to the middle of the edge emanating v_i to v_{i+1}. The remaining two edges of T_i correspond naturally to the midpoint of the remaining edges of the hexpath $\mathcal{H}$ emitting from the vertex v_i .

Definition 3.1.5. *[20] Let $\Theta(X)$ be the point in the plane corresponding to X under the angular correspondence so there is a real affine transformation D of the plane such that $(M(e), N(e)) = R(\Theta(e))$.*

3.2 Billiards path and defining function

3.2.1 Defining Functions

Definition 3.2.1. *[20] Let v_1, v_2 be any two points in $\mathbb{R}^2$, then we write:*

- $v_1 \uparrow v_2$ *if the y-coordinate of v_1 is greater than the y-coordinate of v_2.*
- $v_1 \downarrow v_2$ *if the y-coordinate of v_2 is greater than the y-coordinate of v_1.*
- $v_1 \updownarrow v_2$ *if the y-coordinate of v_1 equals the y-coordinate of v_2.*

The idea of this section is to define a function f_{uv} such that $f_{uv} > 0$ if and only if $u \uparrow v$, and f_{uv} is a function of $(x, y) \in \Omega$ [20].

To prove that a region $Q \subset O(W)$ we just have to prove that $f_{a_i,b_j} > 0$ that is $a_i \uparrow b_j$ for all points (a_i, b_j) in the region Q.

Definition 3.2.2. *[20] Let $\tilde{U}(W,T)$ be the bi-infinite periodic continuation of $U(W,T)$. The image of an infinite periodic polygonal path created in $\tilde{U}(W,T)$ from edges of type d in $U(W,T)$ for each $d \in \{1,2,3\}$ is called the d-spine.*

Denote the complete and irredundant list of edges of the d-spine by $e_1, ..., e_n$, and label the edges such that e_1 is the leftmost edge in the d-spine. Let

$$g_d(x,y) = \Sigma_{k=1}^{n}(-1)^{k-1}exp(iM(e_k)x + N(e_k)y) \tag{3.4}$$

Definition 3.2.3. *[20] Suppose that v_1, v_2 be two vertices of the unfolding $U(W,T)$, then v_1, v_2 are called d-connected if there is a polygonal path of edges of type d joining v_1 to v_2, and d is as large as possible.*

Suppose the list $e_1^{'}, ..., e_m^{'}$ be the edges of type d joining v_1 to v_2 ordered from left to right

$$h(x,y) = \Sigma_{k=1}^{m}(-1)^{k-1}exp(i(M(e_k^{'})x + N(e_k^{'})y)) \tag{3.5}$$

U(W,T) will be rotated so that the first edge is vertical (i.e U(W,T) is in first position):

- The translation direction of $U(W,T)$ is parallel to $\pm ig(x,y)$.
- The vector pointing from p to q is parallel to $\pm h(x,y)$.

So the defining function

$$f(x,y) \pm Im(\bar{g}h)$$

vanishes if and only if $p \updownarrow q$. Here we have set $g = g_d$.

3.3 Computing the turning pairs

As mentioned before the angular correspondence $R(\Theta(e))\Theta$ allows us to find the turning pairs $M(e), N(e)$ by coordinatizing the plane (i.e $R(\Theta(e)) = (M(e), N(e))$). McBilliards computes the turning pairs automatically, by using the unfold window.

3.3.1 Step 1 :Triples

Denote the first digit of W by d. Suppose that $d_{-} \in \{1,2,3\}$ be the congruence class of $d-1$ mod 3. And denote the congruence class of $d+1$ mod 3 by $d_{+} \in \{1,2,3\}$, and set $d_0 = d$, and suppose that $\epsilon \in \{-1,0,1\}$. Define $\alpha_0(d_\epsilon) = \epsilon$ [20].

Suppose that d is the i-th digit of W, and we computed $\alpha_{i-1}(1), \alpha_{i-1}(2), \alpha_{i-1}(3)$ then

$$\alpha_i(d_\epsilon) = \alpha_{i-1}(d_\epsilon) + (-1)^i$$

This way allows us to compute the triple of labels for every triangle in $U(W,T)$. Using McBilliards you can compute these triples from the unfolding window if you click on a triangle of the unfolding. Suppose the plane is coordinatized by the three variables (x,y,z) satisfying the condition that is $x+y+z=0$, so the triple corresponding to T_1 is the coordinates of $\Theta(T_i)$, the ith vertex of the hexpath [20].

3.3.2 Step 2 : Edges

Suppose e is an edge of $U(W,t)$ and e is the dth edge of T_i. Define

$$\beta(e, d_\epsilon) = \alpha_i - (-1)^i \epsilon$$

If T_i is the reflection of the triangle T_{i-1} across e, then e can also be an edge of a T_{i-1} triangle in the unfolding U(W,T) (i.e d is the i-th digit of W).

Denote the leftmost edge of the unfolding $U(W,T)$ by $\mathbf{e} = e(a_1, b_1)$[20].

3.3.3 Eliminating the third angle

By lemma 2.13 in [20] it has been shown that

$$\theta(e) - \theta(\mathbf{e}) = -\frac{\beta(e,1)x + \beta(e,2)y + \beta(e,3)z}{3}$$

In this subsection We will introduce formulas which does not contain the z angle. Define

$$M(e) = \frac{\beta(e,3) - \beta(e,1)}{3}; N(e) = \frac{\beta(e,3) - \beta(e,2)}{3}.$$

Because $z = (-x - y) mod \pi$ so we have

$$\theta(e) - \theta(\mathbf{e}) = M(e)x + N(e)y$$

3.4 The Varification Algorithm

Let P_i be a convex dyadic rational polygon, and let $O(W_i)$ be the orbit tile of W_i. The aim of this algorithm is to show that $P_i \subset O(W_i)$. This method works only for $i = 30, ..., 221$ by producing a cover of P by convex dyadic squares $P \subset \cup Q_i$, where $Q_i \subset O(W)$ for every i. The dyadic rational square is a square in Ω with sides parallel to the coordinate axes and whose vertices have the form $x(\frac{\pi}{2})$ here x is a dyadic rational which belongs to $[0, 1]$ (Where a dyadic rational is a rational number whose denominator is a power of 2).

3.4.1 Certificates of Positivity

Suppose Q is a dyadic rational square, and denote its center by q and its radius by r, r denotes half the edge length of a dyadic rational square Q. Let f be a defining function for a pair of vertices of the unfolding $U(W, T)$. To varify that $f > 0$ on Q by using either the gold or the silver methods. We will mention each method in details.

The Gold Method

In this section we explain the gold method which is described in [20].

Let $\nabla f = (f_x, f_y)$ be the gradient since $f(x, y) = \pm Im(\bar{g}h)$ we have

$$(\bar{g}h)_a = \bar{g}_a h + \bar{g}h_a, \quad a \in \{x, y\}$$

$$f_a = Im(\bar{g}_a h + \bar{g}h_a), \quad a \in \{x, y\}$$

Let

$$f(x,y) = \Sigma_k J_k sin(A_k x + B_k y), \qquad J_k \in N; A_k, B_k \in \mathbb{Z}$$

to be a second form of the defining function. Using a and b instead of x and y, the second derivatives of f have bounds

$$|f_{ab}| \leq F_{ab}$$

Since

$$f_x = \sum_k J_k A_k Cos(A_k x + B_k y)$$

$$f_{xx} = -\sum_k J_k A_k^2 sin(A_k x + B_k y)$$

$$| f_{xx} | \leq \sum_k | J_k || A_k |^2 | sin(A_k x + B_k y) |;$$

$$| f_{xx} | \leq \sum_k |J_k| A_k|^2.$$

So,

$$F_{xx} = \sum_k A_k^2 |J_k|$$

$$f_{xy} = -\sum_k J_k A_k B_k sin(A_k x + B_k y)$$

$$|f_{xy}| \leq \sum_k |J_k||A_k||B_k||sin(A_k x + B_k y)|$$

$$|f_{xy}| \leq \sum_k | J_k || A_k || B_k | .$$

So,

$$F_{xy} = \sum_k A_k B_k |J_k|$$

$$f_y = \sum_k J_k B_k cos(A_k x + B_k y)$$

$$f_{yy} = -\sum_k J_k B_k^2 sin(A_k x + B_k y)$$

$$| f_{yy} | \leq \sum_k | J_k | |B_k|^2.$$

Therefore,

$$F_{yy} = \sum_k B_k^2 |J_k|.$$

Let r be of the form

$$r = \frac{\pi}{2} x$$

where x is some dyadic rational number. Define the quantities

$$a_x = r(F_{xx} + F_{xy}) \quad ; \quad a_y = r(F_{yx} + F_{yy})$$

Finally, define the rectangle

$$G(q, f) = [f_x(q) - a_x, f_x(q) + a_x] \times [f_y(q) - a_y, f_y(q) + a + y]$$

then

$$\nabla f(x, y) \subset G(Q, f) \qquad ; \quad \forall (x, y) \in Q.$$

When f is gold certified, then $G(q, f)$ is completely contained in one of the standard quadrants in $\mathbb{R}^2$. There exist a vertex v on Q such that ∇f is a positive linear combination of the edges of Q when f is gold certified. That is $f(x, y) > f(v)$ foe every $(x, y) \in Q$. So if f is gold certified and $f(v) > 0$ then $f|_Q$ is also greater than zero. This is called the gold method to prove $f|_Q > 0$.

The Silver Method

In [20] Schwartz describes the following algorithm called silver method, In this subsection we introduce this method which is defined as follows:

Define a not dyadic rational square $\hat{Q}$ such that Q is midscribed in $\hat{Q}$. While its vertices and dyadic square have the form πx with x a dyadic rational, as shown in the figure below

Define the rectangle

$$S(q, f) = [f_x(q) - 2a_x, f_x(q) + 2a_x] \times [f_y(q) - 2a_y, f_y(q) + 2a_y]$$

With the same notation as in the gold method. then

$$\nabla f(x, y) \subset S(Q, f); \forall (x, y) \in \hat{Q}.$$

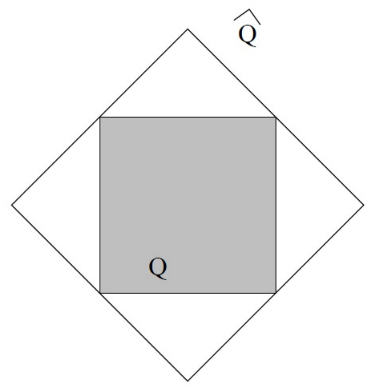

Figure 3.1: Q is midscribed in $Q^{'}$.

When f is silver certified, then $S(q, f)$ is completely contained in one of the standard quadrants in $\mathbb{R}^2$ after rotating them by an angle 45°. There exist a vertex v of $\hat{Q}$ such that ∇f is a positive linear combination of the edges of $\hat{Q}$ when f is silver certified, in other words it is disjoint from the lines which passes through the origin with a slope ± 1. When f is silver certified, then there exist a vertex v of $\hat{Q}$ such that ∇f within $\hat{Q}$ is a positive linear combination of the edges of $\hat{Q}$ that means $f(x, y) > f(v)$ for every $(x, y) \in \hat{Q}$ and it is also hold for $(x, y) \in Q$ so if f is silver certified and $f(v) > 0$ then $f|_Q > 0$ is also greater than zero.

A Technical Point

When we will compute rigorous computation r in the above formula will be replaced by the larger

$$\tilde{r} = 2x$$

Then we will work with the rectangles $\tilde{G}(Q, f)$ and $\tilde{S}(Q, f)$ with the same definition as above but r is replaced by $\tilde{r}$. This replacement helps us to reduce the problem to an integer calculation. However it makes the functions a bit harder to certify.

3.4.2 An Inefficient First try

In this subsection we introduce a simple first try algorithm written by schwartz in [11] which is so slow. This algorithm goes as follows.

Suppose that Q is a dyadic square and W is a word. If for each defining function f_{ij}, we can show $f_{ij}|_Q > 0$ by the gold or silver method, then W is called good with respect to Q. When $Q \subset O(W)$ when W is good. Suppose that $Q_0 = [0, \frac{\pi}{2}]^2$. Start with a list of squares, with Q_0 as its sole member. Suppose that Q is the last square on the list of dyadic rational square. There are many choices:

- When f is good in Q, delete Q from the list and add it to the covering.
- When $Q \cap P = \phi$, then delete Q from the list.
- If the two cases above does not hold, then Subdivide Q in half to get four squares and replace Q by them.

P will be covered by dyadic squares, when the list becomes empty, that contained in O(W). If the list becomes empty, then we have a covering of P by dyadic squares, which contained in $O(W)$. The problem of this algorithm is that it is too slow.

3.4.3 The Tournament

Let W and Q be given such that W is a fixed word and Q is a dyadic rational square. Let A and B be two lists of indices, such that A is a list of distinct a-vertices, and B is a list of distinct b-vertices.

Definition 3.4.1. *[20] Two indices i and $j \in A$ are said to be adjacent vertices if $i < j$, and there is no index $c \in A$ such that $i < j < c$.*

Definition 3.4.2. *[20] If i and j are two adjacent indices in A, then a defining function corresponded to (a_i, a_j) is called an* ***A - function****.*

Definition 3.4.3. *[20] A vertex $i \in A$ is called an A-loser if one of the following two conditions holds:*

- *Suppose i and j are two adjacent indices with $j > i$, and if f is an A-function corresponding to (a_i, a_j). Then by using the silver or the gold method $-f_Q$ is a positive.*
- *Suppose i and j are two adjacent indices, with $j < i$ and if f is an A-function associated to(a_i, a_j), then f_Q can be certified positive.*

When i is the first or last index in A, then one of the following condition will not hold. And If i is the only index in A then the two conditions above will not hold.

If $i \in A$ is an A-loser then there exist an index $j \in A$ such that $a_i \uparrow a_j$ throughout Q, so if $a_j \uparrow b_k$ within Q, then we conclude that $a_i \uparrow b_k$ in Q. When i is not an A-loser then it is an A-survivor.

With the B list all the definitions in the A-list hold except the sign of the defining function.

- Suppose i and j are two adjacent indices with $j > i$, and if f is an B-function corresponding to (b_i, b_j). Then by using the silver or the gold method f_Q is a positive [20].
- Suppose i and j are two adjacent indices, with $j < i$ and if f is an B-function associated to(b_i, b_j), then $-f_Q$ can be certified positive [20].

Let $f_1, ..., f_m$ be a list of A-functions. Create a new list A' of the A-survivor. When $A' = A$ then A is called stable. If not, form a sequence $A \supset A' \supset A'' \supset$ ..... until the list stabilizes.

This procedure is called an A-tournament on Q. The indices of the last list is called the A-winners [20].

3.4.4 The improved algorithm

The improved algorithm written by Schwartz in [20] which goes as follows.

Begin this improved algorithm with the list (Q_0, A_0, B_0). Here Q_0 as mentioned before. And A_0,B_0 are are called the complete set of indices, such that $A_0 = B_0 = \{1, 2, ..., k\}$, where k equals half length of W. Let (Q, A, B) be the final triple on the list at any step.

When $Q \cap P = \phi$, delete $(Q, A, B)$from the list and continue. Else.....

- Perform the A -tournament and B- tournament to obtain (Q, A^*, B^*). Where A^* contains the A -winners and B^* contains the B-winners.

- Let i and j be any elements in A^* and B^* respectively, such that $(i, j) \in (A^*, B^*)$. We will try to prove that $f_{ij} \mid_Q > 0$. Add Q to the covering of P when we succeed for each pair of P. Else ....

- Delete the triple (Q, A, B) from the list and bisecting it in to four triples (Q_j, A^*, B^*),$j = 1, .., 4$, and replace (Q, A, B) by this four triples.

The covering of dyadic square of P will be obtained and contained in O(W), if the list becomes empty.

Lemma 3.4.1. *[20] If Q is added to our cover then* $Q \subset O(W)$

3.5 McBilliards

In this section we give some features about Mc-Billiards programm which is written by Schwartz and hooper in [13].

3.5.1 The Plotting Console and Color Selector

When McBilliards is ready to plot, it has two pairs of information: the word W and a point $(x, y) \in Tile(W)$. McBilliards then keeps track of an angle θ beginning with $\theta = 0$. McBilliards has 4 different plot options:

- The basic plot: McBilliards finds the intersection of the ray R_θ with the boundary of $Tile(W)$. This intersection point is plotted and then θ is incremented. Here R_θ is the angle that the ray makes with the positive x-axis. If the number of data points is N then θ is incremented by $\frac{2\pi}{N}$ and a total of N points are used to plot the tile. The numbered buttons let the user change the value of N. When the value of N is higher, then the plots is sharper [13].

- newton plot: McBilliards finds the vertices of the orbit tile using Newton's method, and then plot the edges between the vertices by using Newton method [13].

- Convex hull: By Newton's Method McBilliards find the vertices, then takes the polyogn spanned by these vertices. In practice his polygon is always convex, and hence coincides with convex hull of the vertices. The convex hull of the vertices usually contains the tile as a proper subset [13].

- inner hull: After computing the polygon spanned by the N vertices, McBilliards inserts a new vertex near the center of each edge, producing a 2N -gon which seems always to be a proper subset of the tile [13].

3.5.2 Button 2 controls

The Buttton 2 controls determine the behavior of the middle mouse button when it is clicked on parts of McBilliards. Three of the buttons have to do with scaling and scrolling. The remaining buttons have to do with the modification of the plotted orbit tiles. Each tile can be

1. Simply recognized. In the "normal" mode of function, clicking a tile simply focuses McBilliards' attention on this tile. For instance, the word corresponding to the tile is drawn and the unfolding relative to the word and the selected triangle is drawn [13].
2. deleted;
3. recolored;
4. raised relative to other tiles;
5. lowered relative to other tiles;
6. Cycled - that is the unfolding for the word is replaced by the unfolding for the left rotation of the word;
7. recentered. When the tile is initially drawn, some center point (x,y) is used. This center point can be changed [13].

   There is one additional option which have called slop. This option tells McBilliards to draw the unfolding U(W,T) even when the selected triangle T does not correspond to a point of Tile(W). On other words, the unfolding feature is allowed to "slope over" the edges of the tile.

3.5.3 The Postscript Window

This window allows the user save pictures of each of the three main windows in to post script files

- The parameter window
- The unfolding window
- The plotting window

3.5.4 The Labelling Console

The small window marked lets the user to to add and delete labels to the parameter or unfolding window. If you click on the window, McBilliards focuses the attention of the keyboard to the strip. Then the user can enter a label [13].

3.5.5 The Rational Overlay Console

The rational overlay console allows the user to locate specific rational points in the parameter window, by moving a crosshairs around in rational jumps. The cluster of buttons on in the upper right hand corner lets the user to move a crosshairs around the parameter window. The button in the northwest corner moves the cross in the northwest direction, and so forth. The cross moves by rational jumps [13] .

The step button controls the size of the jumps. When the dyadic option is selected, the jump go by way of dyadic fraction. In this situation, a stepsize of N causes the jumps to be of size 2^{-N} [13].

When the Farey option is selected, the jumps move from one level -N Farey fraction to the next one. The first few levels of Farey fraction are :

$$\frac{0}{1}; \frac{1}{1}$$

$$\frac{0}{1}; \frac{1}{2}; \frac{1}{1}$$

$$\frac{0}{1}; \frac{1}{3}; \frac{2}{3}; \frac{1}{1}$$

$$\frac{0}{1}; \frac{1}{4}; \frac{1}{2}; \frac{3}{4}; \frac{1}{1}$$

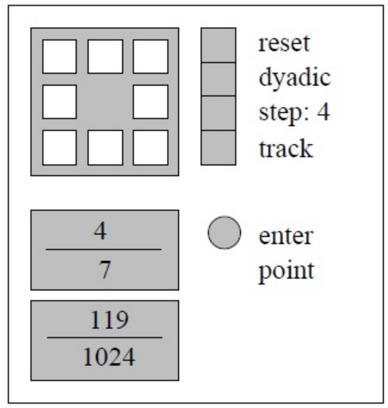

Figure 3.2: The rational overlay console.

In general, level N is obtained from level N-1 by using Farey addition law on consecutive entries, similar to Pascal's triangle. Every rational numbers appears at some level. The level of $\frac{p}{q}$ is the number of steps in the Euclidean algorithm applied to (p, q). The Farey option is an efficient way of accessing all rational numbered [13].

When activated, the track button allows the user control the location of the crosshairs by directly clicking on the parameter window. McBilliards then picks the best rational approximation to the selected point, according to the stepsize .

The two rectangles at lower left display the coordinates of the cross. The enter point button allows the user select preciselt this points on the parameter window.

3.5.6 The triangle Entry Console

The triangle entry console allows the user triangule portions of the parameter window by using triangle of rational coordinates . First, select the rational coordinates by the rational overlay console. When the rational coordinate has bee selected. Mcbilliards plots the determined triangle by corresponding to each rational number to the corresponding vertex when the user click on one of the vertices of the central triangle in the triangle entry console[13].

McBilliards decide which triangle to determine with the current mouse click, when the user click on the triangle button.

3.6 The Basic Search Algorithm

3.6.1 Balanced Words and Orbit Tiles

U(W,T) is the unfolding of a triangle T with a corresponding word W.

Definition 3.6.1. *[13] The word W is balanced if the first and last sides of U(W,T) are parallel for every triangle T.*

As mentioned before we break W in to couplets. And we let N_{ij} denote how many times the couplet ij appears in the given word.

If $U(W,T)$ has a centerline let Tile(W) to be the set of all triangles $T^{'}$ such that $U(W,T^{'})$ has a centerline. Then Tile(W) is an open subset of the parameter space of triangles.

3.6.2 The Weak Test

The weak test is a computational test applied to the pair (W,T) by Hooper and Schwartz in [13]. The algorithm of this test goes as follows:

- Let v_1 and v_2 be the two endpoints of one of the two special paths .
- Check the sign of the signed area of the triangle $\triangle(v_1, v_2, w)$ for every vertex w on the other path.
- Reverse the roles of the two paths and repeat.

When we obtain the wrong sign always, then we conclude that When(W,T) fails the weak test. Therefore if $Tile(\hat{W},T) = \phi$ for every balanced word $\hat{W}$ that contains W as a subword [13].

3.6.3 The Strong Test

In this subsection we introduce the strong test written by Hooper and Schwartz in [13].

The strong test checks that W is balanced. If W is not balanced then W fails the strong test. Henceforth assume W is balanced .

There is two special paths of U(W,T) joining v_1 to v_2. Suppose $v_1', ..., v_n'$ be the remaining vertices of one of these paths. Define

$$m' = maxArea(\Delta(v_1, v_2, v_j')).$$

Define the remaining vertices of the other special path of U(W,T) by $w_1, w_2, ..., w_m$, And define

$$m = minArea(\triangle(v_1, v_2, w_j))$$

If you rotate the graph such that the translation taking the first edge to the last edge of $U(W,T)$ is horizontal, then the areas we have computed are all proportional to the y-coordinates of these vertices. The constant of proportionality is independent of vertex. Thus U(W,T) has a lane (centerline) if and only if $m' < m$. Thus (W,T) passes the strong test if and only if W is the orbit type of a periodic billiard path on T.

The Lexi Test The lexi-test is a test which is applied to words W. A word W fails the lexi-test if and only if W contains a subword W' which comes before W in the lexicographic ordering. For example, $W = 312131$ fails the lexi-test because according to the lexicographic ordering the subword $W' = 2131$ comes before W. The search algorithm throws out words which fail the lexi-test because they are redundant. Because of throwing out McBilliards does not find all possible periodic orbits of even length up to certain depth. Rather, McBilliards find all possible equivalence classes of orbit types, where two types are equivalent if they are rotations of each other [13].

3.6.4 The Algorithm

Denote the depth of the search by D, and a triangle by T. Start the search algorithm by a list of words called **CONTENDERS** and another list of words called **WINNERS**. Initially **CONTENDERS** contains the single word 12 and the second list is empty. The algorithm proceeds until the first list is empty, then cuts. Now, winners is the list of even length balanced words of length less than or equal to D which are orbit types of periodic billiard paths in T [13].

1. If CONTENDERS $\neq \phi$ let W be the first word on CONTENDERS .

2. If W fails the lexi test or the weak test, delete W from CONTENDERS and return to step 1.

3. If W passes the strong test append W to WINNERS.

4. Let L = Length (W). If $L \leq D-2$ then delete W from CONTENDERS and prepend to CONTENDERS the four words W_1, W_2, W_3, W_4 which have length $L+2$ and contains W as its initial word. Go to step 1 [13].

For example, if we take the word W = 13, then the four words are 1313, 1312, 1321, and 1323.

The algorithm implements a depth first search by the tree of words pruning off any branches whose initial node fails the weak test or lexi test [13].

Remark 4. *[13] The right angled search works just as the balanced search, except that the balance condition is different. Here we weaken it to the condition that the difference $N_{j3} - N_{3j}$ is independent of $j = 1, 2$ and $N_{12} - N_{21}$ is even.*

3.7 An application of calculus to triangular billiard

Two applications of billiards given by Gutkin [10]. In this section we will recall one of them.
Let T_1 denotes the pedal triangle of a given triangle T which is formed by the feet of the three altitudes of T. Suppose Υ be the space of all triangles. Then $p : T \rightarrow T_1$ is a natural self-mapping of Υ, the pedal triangle T_1 is the shortest such orbit and it is the only closed (prime) billiard orbit known .

Two triangles T and $T^{'}$ are close in the Euclidean topology if we label the vertices of T=ABC and $T^{'} = A^{'}B^{'}C^{'}$ such that A is close to $A^{'}$, and B is close to $B^{'}$, C is close to $C^{'}$. Denote the subspace of acute triangles by $\mathcal{F}$. The relative length of the fagnano orbit $f(t) = \frac{|T_1|}{|T|}$, is a positive continuous function on $\mathcal{F}$. Let $|P|$ denote the perimeter of a polygon. Then harmonic polygons are the critical points of the function $|p|$ on the space of polygons. inscribed in T. For any convex C^1 billiard table, this fact is crucial for the existence of periodic orbits suppose p and q are positive integers. A periodic orbit with q sides that goes p times around the table has $\frac{p}{q}$ rotation number, and period q. For any rational number, $0 < \frac{p}{q} < 1$, with p and q relatively prime, there are at least two distinct periodic orbits of period q, with rotation number $\frac{p}{q}$ which is called the Birkhoff periodic orbits. The condition that the table is C^1 is necessary for the preceding assertion.

For example, there is no triangular table which has a periodic orbit of period 2 and with rotation number $\frac{1}{2}$ and obtuse triangle have no periodic orbits of period 3 and with rotation number $\frac{1}{3}$.

Theorem 3.7.1. *[10] The maximal relative length of the Fagnano periodic orbit is $\frac{1}{2}$. It is attained at the equilateral triangles.*

Bibliography

[1] Baxter,A.M., "Combinatorial properties of billiards on an equilateral triangle." arxiv preprint math/00509292. 2005

[2] Baxter,A.M and Umble,R. "Periodic Orbits of billiards on an equilateral triangle." 2007.

[3] Boothby,W. "An introduction to Differentiable manifolds and Riemann geometry." A subsidiary of Hrcourt Brace Jovanovich. 1975.

[4] Brooks,D. "N immpact trajectories in rectangular billiards." 2011

[5] Clarkson,C. "Classification of Surfaces." 2006.

[6] Climenhaga,V. "Lectures on surfaces Surfaces." 2008.

[7] Contini, S. "Greatest Common Divisor." Encyclopedia of Cryptography and Security, (2011): 518-519.

[8] Davis,D. "Billiard and flat surface." Mathematisches Forschurgsinstitut. 1(2015).

[9] Galperin,G and Zvonkine,D., "periodic billiard trajectories in right triangles and right angled tetrahedra." Regular. Chaotic Dyn. 8.1(2003): 29-44.

[10] Gutkin,E. "Two application of calculus to triangular billiards." The American mathematical monthly. 104.7(1997): 618-622

[11] Harris,G. "Polygonal Billiards." Southern Illinois university Edwardsville. 2007.

[12] Hooper,W.P., "Dynamics on an infinite surface with the lattice property." arxiv preprint: 0802.0189. 2008.

[13] Hooper,W.P and Schwartz,R.E. "Introduction to McBilliards." preprint. 2004.

[14] Hooper,W.P., "Periodic billiard path in right triangles are unstable." Geometriae Dedicata. 125.1(2007):39-46.

[15] Katok,A. "The growth rate for the number of singular and periodic orbits for a polygonal billiards." Communications in Mathematical physics. 111.1(1987): 151-160.

[16] Marsden,R. "Introduction to Machanies and symmetry ",2^{nd} edition, springer. 1998: 121-146.

[17] Masur,H., Tabachnikov, S. "Rational billiards and flat structures." MPI. 1999.

[18] Schlag,W. "A concise course in complex analysis and Riemann surfaces." Vol.154(2014): (65-66).

[19] Schmurr, J. "Translation covers among triangular billiard surfaces." Illinois Journal of Mathematics, 58.3(2015): 665-687.

[20] Schwartz, R.E. "Obtuse triangular billiard 2 :100 degree worth of periodic trajectories." Experimental Mathematics. 18.2(2006):137-171.

[21] Schwartz, R.E., "Billiards obtuse and irratioal." Preprint, 2005

[22] Troubetzkoy, S. "Periodic billiard orbits in right triangles." In Annales delinstitut Fourier. Vol.55. No.1(2004): 29-46.

[23] Troubetzkoy, S. "Periodic billiard orbits in right triangle 2." 2005

[24] Tabachnikov, S. "Geometry and Billiards." American Mathematical society. Vol.30(2005).

[25] Troubetzkoy, S. "Aproximation and billiards." arxiv preprint math/0611599. (2007).

[26] William, S.M., "Algebraic topology: An introduction (Harcourt Brace and World)". 1967.

[27] Yang, Y. "Stable billiard path on polygons." 2013.